9/31/57 ggl

AFOSR TN-57-130
ASTIA DOCUMENT No. AD 120 486

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. BR-21

The Asymptotic Theory of Solutions of $\Delta u + k^2 u = 0$

WILLARD L. MIRANKER

Mathematics Division
Air Force Office of Scientific Research
Contract No. AF 18(600)-367
File No. 2.2
APRIL, 1957

BR-21

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3,

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. BR-21

THE ASYMPTOTIC THEORY OF SOLUTIONS OF $\Delta u + k^2 u = 0$

Willard L. Miranker

Willard L. Miranker
Willard L. Miranker

Morris Kline
Morris Kline
Project Director

April, 1957

Qualified requestors may obtain copies of this report from the ASTIA Document Service Center, Dayton 2, Ohio. Department of Defense contractors must be established for ASTIA services, or have their 'need-to-know' certified by the cognizant military agency of their project or contract.

New York, 1957

Abstract

The subject of this report is the asymptotic theory of solutions, u, of the reduced wave equation, $\Delta u + k^2 u = 0$, defined in infinite domains.

In Section 1 we furnish new proofs of three well-known theorems concerning u. These are Rellich's growth estimate, the uniqueness theorem for the exterior boundary-value problem, and the representation theorem. A new result, the representation theorem for u when the boundary of the domain of definition of u is infinite, is also given.

In Section 2 Rellich's growth estimate is extended to solutions of the equation $\Delta v + k^2(x)v = 0$. From this result we are able to deduce various uniqueness and representation theorems for solutions of this equation.

In Section 3 we show that the normal boundary values of a radiating solution, u, of $\Delta u + k^2 u = 0$ is bounded by a homogeneous quadratic functional of its boundary values. This result combined with the representation theorem for u yields an L^2-maximum principle for u.

Finally, in Section 4 the behavior of u when the parameter k becomes large is considered. We explain the method of G. Birkhoff for obtaining formal asymptotic expansions for u, and deduce several results concerning the existence and validity of these formal expansions.

Table of Contents

		Page
1.	Uniqueness and representation theorems for solutions of $\Delta u + k^2 u = 0$	1
2.	The reduced wave equation in a medium with a variable index of refraction	9
3.	The L^2-maximum principle	19
4.	Parametric theory of solutions of the reduced wave equation $\Delta u + k^2 u = 0$	30
References		47

1. Uniqueness and representation theorems for solutions of $\Delta u + k^2 u = 0$

In 1943 F. Rellich[14] deduced the so-called growth estimate for solutions of the reduced wave equation

$$\text{(A)} \qquad \Delta u + k^2 u = 0.$$

This estimate applies to solutions defined in the exterior D of a bounded domain. Using this estimate, he showed that u is identically zero if it is zero on the boundary B of D and satisfies the radiation condition.

Here we present a new proof of the growth estimate which seems to be simpler than Rellich's original proof. The method of our proof is similar to one which is used in deriving the mean value theorem for solutions of (A) (see [15]). By means of the same method, we obtain the well-known theorem concerning the representation of u as a sum of single and double layers. Our proof of this theorem has the advantage that, apart from the radiation condition, it makes no use of asymptotic properties of u. The method is also used to obtain a new result, namely a representation theorem for u in the case where the boundary B of D is infinite.

The idea of representing a solution of (A) as a sum of single and double layers dates back at least to Poincaré. The formal result appears in Sommerfeld's 1912 paper[16]. The proof of this result follows trivially from the radiation and boundedness conditions when B is bounded (see e.g., [1]). Since these conditions were introduced by Sommerfeld, the proof of the representation theorem was probably known to him in 1912. In 1948 W. Magnus[11] proved that the radiation condition implies the boundedness condition when B is bounded. Thus implicitly contained in Magnus' paper is a proof of the representation theorem for bounded B using only the radiation condition. An explicit proof may be found in C. Wilcox[21](1956).

With the exception of boundaries of certain special shapes, the representation theorem has not hitherto been proved for unbounded boundaries.

We first state the results in the form of theorems; all the proofs are given later.

Let D be the complement of a bounded domain with boundary B. Let $\sum_R$ be the surface of a sphere of radius R and let B be such that Green's theorem can be applied to $D_R = D \cap (\text{interior of } \sum_R)$ for $\sum_R$ containing B. Let u be a solution in D of

$$(1.1) \qquad \Delta u + k^2 u = 0.$$

The radiation condition is

$$\lim_{R \to \infty} \iint_{\Sigma_R} \left| \frac{\partial u}{\partial n} - iku \right|^2 dS = 0; \tag{1.2}$$

dS denotes a surface element and dV a volume element. The letters P and Q denote points in D. For this situation we will prove the following theorems.

Theorem 1.1 (Rellich's growth estimate). Let u satisfy (1). Then

$$\lim_{R \to \infty} \iint_{\Sigma_R} |u|^2 \, dS \neq 0 \tag{1.3}$$

unless u is identically zero.

Lemma 1.1. Let u satisfy (1) and (2), and let u or $\frac{\partial u}{\partial n}$ vanish on B. Then

$$\lim_{R \to \infty} \iint_{\Sigma_R} |u|^2 dS = 0.$$

Theorem 1.2 (uniqueness theorem). There exists at most one solution of (1) which satisfies (2) and has prescribed values or prescribed values of its normal derivative on B.

Theorem 1.3 (representation theorem). Let u be a solution of (1) which satisfies (2). Then

$$u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS. \tag{1.4}$$

Corollary. Let u be a solution of $\Delta u + k^2 u = f$ which satisfies (2) and for which

$$\iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} \, dV$$

exists. Then

$$u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS - \frac{1}{4\pi} \iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} \, dV.$$

Now let D be the complement of a domain whose boundary B is unbounded. Let $D_R = D \cap$ (interior of Σ_R), let $B_R = B \cap$ (interior of Σ_R), and let $E_R = D \cap \Sigma_R$. Let A(R) be the area of E_R. Let B be such that we can apply Green's theorem to D_R. The radiation condition becomes

$$\lim_{R \to \infty} \iint_{E_R} \left| \frac{\partial u}{\partial R} - iku \right|^2 dS = 0. \tag{1.5}$$

For this situation we will prove the following theorems.

Theorem 1.4 (representation theorem). Let u be a solution in D of (1) which satisfies (5). Let the following integrals exist:

$$\iint_B \left(\frac{\cos k|P-Q|}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\cos k|P-Q|}{|P-Q|} \right) dS$$

$$\iint_B \left(\frac{\sin k|P-Q|}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{\sin k|P-Q|}{|P-Q|} \right) dS \,.$$

Then

$$(1.6) \qquad u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS.$$

Remark: These integrals exist if the values of u on B and the values of the normal derivative of u on B satisfy the boundedness condition

$$(1.7) \qquad \lim_{R \to \infty} \left(R|u|,\ R\left|\frac{\partial u}{\partial n}\right| \right) < \text{constant}.$$

Corollary. Let u be a solution of $\triangle u + k^2 u = f$ which satisfies (5). Let the integrals

$$\iiint_D f(Q) \frac{\sin k|P-Q|}{|P-Q|} dV$$

and

$$\iiint_D f(Q) \frac{\cos k|P-Q|}{|P-Q|} dV$$

exist. Then under the hypothesis of Theorem 4,

$$u(P) = \frac{1}{4\pi} \iint_B \left(\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \frac{e^{ik|P-Q|}}{|P-Q|} \right) dS$$
$$- \frac{1}{4\pi} \iiint_D f(Q) \frac{e^{ik|P-Q|}}{|P-Q|} dV.$$

Theorem 1.5. Let u be a solution in D of (1) which satisfies (5). Let u = 0 on B and let the integrals which appear in (8) and (9) below exist. Then

$$(1.8) \qquad u(P) = \frac{1}{4\pi} \iint_B \frac{\cos k|P-Q|}{|P-Q|} \frac{\partial u}{\partial n} dS$$

and

$$(1.9) \qquad 0 = \frac{1}{4\pi} \iint_B \frac{\sin k|P-Q|}{|P-Q|} \frac{\partial u}{\partial n}\, dS.$$

All the results of this section are valid in two dimensions, if we replace $\frac{1}{4\pi} \frac{e^{ik|P-Q|}}{|P-Q|}$ and its real and imaginary parts respectively by $-\frac{i}{4} H_o^{(1)}(k|P-Q|)$ and its real and imaginary parts. A similar remark applies to dimensions greater than three.

We now give the proofs of all these theorems.

Proof of Theorem 1.1. Let P be the center of $\sum_R$. Let P be outside of B and let $\sum_R$ contain B. Then apply Green's theorem

$$(1.10) \qquad \iiint (u \triangle v - v \triangle u) dV = \iint \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n}\right) dS$$

to D_R. First let v be $\frac{\cos k|P-Q|}{|P-Q|}$ and then $\frac{\sin k|P-Q|}{|P-Q|}$. If ρ is the distance from P to the point of integration on B, we obtain for these two v

$$(1.11) \qquad \begin{aligned} u(P) = {} & \frac{1}{4\pi} \iint_{\sum_R} \left[\frac{\cos kR}{R} \frac{\partial u}{\partial R} - u \frac{\partial}{\partial R} \left(\frac{\cos kR}{R}\right)\right] dS \\ & + \frac{1}{4\pi} \iint_B \left[\frac{\cos k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\cos k\rho}{\rho}\right)\right] dS \end{aligned}$$

and

$$(1.12) \qquad \begin{aligned} 0 = {} & \frac{1}{4\pi} \iint_{\sum_R} \left[\frac{\sin kR}{R} \frac{\partial u}{\partial R} - u \frac{\partial}{\partial R} \left(\frac{\sin kR}{R}\right)\right] dS \\ & + \frac{1}{4\pi} \iint_B \left[\frac{\sin k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\sin k\rho}{\rho}\right)\right] dS. \end{aligned}$$

Now multiply (11) by $\frac{\sin kR}{R}$ and (12) by $-\frac{\cos kR}{R}$, and add. We obtain

$$(1.13) \qquad \begin{aligned} \frac{\sin kR}{R} u(P) = {} & \frac{1}{4\pi} \left[\frac{\cos kR}{R} \frac{\partial}{\partial R} \left(\frac{\sin kR}{R}\right) - \frac{\sin kR}{R} \frac{\partial}{\partial R} \left(\frac{\cos kR}{R}\right)\right] \iint_{\sum_R} u\, dS \\ & + \frac{\sin kR}{4\pi R} \iint_B \left[\frac{\cos k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\cos k\rho}{\rho}\right)\right] dS \\ & - \frac{\cos kR}{4\pi R} \iint_B \left[\frac{\sin k\rho}{\rho} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{\sin k\rho}{\rho}\right)\right] dS. \end{aligned}$$

Rewrite (13) as

$$\frac{\sin kR}{R}\left\{u(P) - \frac{1}{4\pi}\iint_B \left[\frac{\cos k\rho}{\rho}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{\cos k\rho}{\rho}\right)\right] dS\right\} \tag{1.14}$$
$$+ \frac{\cos kR}{R}\left\{\frac{1}{4\pi}\iint_B \left[\frac{\sin k\rho}{\rho}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{\sin k\rho}{\rho}\right)\right]\right\} = \frac{k}{4\pi R^2}\iint_{\Sigma_R} u\, dS.$$

We introduce the abbreviations α and β by rewriting (14) as

$$\alpha\frac{\sin kR}{R} + \beta\frac{\cos kR}{R} = \frac{k}{4\pi R^2}\iint_{\Sigma_R} u\, dS; \tag{1.15}$$

α and β are independent of R. Applying Schwarz' inequality to the right member of (15), we obtain

$$\left|\alpha\frac{\sin kR}{R} + \beta\frac{\cos kR}{R}\right| \leq \frac{k}{2\sqrt{\pi}R}\sqrt{\iint_{\Sigma_R} |u|^2\, dS} \quad . \tag{1.16}$$

If α and β are not both zero for all choices of P, then for some P, the left member of (16) is exactly $O(1/R)$ as R tends to infinity. Then (16) yields

$$\lim_{R\to\infty}\iint_{\Sigma_R} |u|^2\, dS > 0. \tag{1.17}$$

In the case where α and β are zero for all choices of P, we must show that $u \equiv 0$. Suppose then that $\alpha \equiv \beta \equiv 0$. Add $\pm i\beta$ to α. We obtain

$$u = \frac{1}{4\pi}\iint_B \left[\frac{e^{\pm ik\rho}}{\rho}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\left(\frac{e^{\pm ik\rho}}{\rho}\right)\right] dS. \tag{1.18}$$

We rewrite (18) as

$$u = u_s + u_d . \tag{1.19}$$

Let r and r' be the radial polar coordinates of P and the point of integration, Q, in (18) respectively. Let O be the origin of these polar coordinates, and let $\cos\chi$ be the cosine of the angle between $\overline{PO}$ and $\overline{QO}$. We have then

$$u_z(P)e^{\mp ikr} = \frac{1}{4\pi}\iint_S \frac{\partial u}{\partial n}\,\frac{e^{\pm i\left[\sqrt{r^2 - 2rr'\cos\chi + r'^2} - r\right]}}{\sqrt{r^2 - 2rr'\cos\chi + r'^2}}\, dS. \tag{1.20}$$

If r is sufficiently large, the quantity under the radicals in (20) cannot vanish since B and therefore r' are bounded. Thus the kernel in (20) is an analytic function of r in a neighborhood of $r = \infty$. Multiplication by the continuous function $\partial u/\partial n$ and integration over B preserves this analyticity. (If $\partial u/\partial n$ is not continuous, replace B by a sphere containing it.) Thus the right member of (20) is an analytic function of r in a neighborhood of $r = \infty$. In a similar manner this same statement holds for $e^{\mp ikr}u_d$. Thus from (19), we have

$$(1.21) \qquad u\, e^{\mp ikr} = A^{\mp}(r)$$

where A^+ and A^- are analytic functions of r in a neighborhood of $r = \infty$.

If u is not identically zero, we may divide one relation in (21) by the other to get

$$e^{-2ikr} = \frac{A^-(r)}{A^+(r)} .$$

But this is absurd since the left member has an essential singularity at $r = \infty$, while the right member has at most a pole there. Thus $u \equiv 0$ and the theorem is proved.

Proof of Lemma 1.1. Let $\bar{u}$ denote the complex conjugate of u. From Green's theorem we have

$$\begin{aligned}\iiint_{D_R} \left[\bar{u}(\triangle u + k^2 u) - u(\triangle \bar{u} + k^2\bar{u})\right]dV &= \iint_B \left[\bar{u}\frac{\partial u}{\partial n} - u\frac{\partial u}{\partial n}\right]dS \\ (1.22) \qquad &+ \iint_{\Sigma_R} \left[\bar{u}\frac{\partial u}{\partial R} - u\frac{\partial \bar{u}}{\partial R}\right]dS.\end{aligned}$$

Since $\triangle u + k^2 u = 0$ and $\triangle \bar{u} + k^2\bar{u} = 0$ in D_R, and $u = \bar{u} = 0$ or $\frac{\partial u}{\partial n} = \frac{\partial \bar{u}}{\partial n} = 0$ on B, (22) reduces to

$$(1.23) \qquad \iint_{\Sigma_R} \left[\bar{u}\frac{\partial u}{\partial R} - u\frac{\partial \bar{u}}{\partial R}\right] dS = 0.$$

Expanding the integrand in the radiation condition (2), we get

$$(1.24) \qquad 0 = \lim_{R\to\infty} \iint_{\Sigma_R} \left[\left|\frac{\partial u}{\partial R}\right|^2 + k^2|u|^2 - ik\left(u\frac{\partial \bar{u}}{\partial R} - \bar{u}\frac{\partial u}{\partial R}\right)\right] dS.$$

Inserting (23) into (24) we obtain

$$0 = \lim_{R\to\infty} \iint_{\Sigma_R} \left[\left|\frac{\partial u}{\partial R}\right|^2 + k^2|u|^2\right] dS,$$

which contains the assertion of the lemma.

Proof of Theorem 1.2. Suppose there are two solutions u_1, u_2, of (1) satisfying (2) and having prescribed values or prescribed normal derivatives on B. Let $u = u_1 - u_2$. An application of the Schwarz' inequality shows that u satisfies the radiation condition, (2). Thus since $u = 0$ or $\partial u/\partial n = 0$ on B, Lemma 1 implies that

$$\lim_{R \to \infty} \iint_{\Sigma_R} |u|^2 \, dS = 0.$$

But then Theorem 1 implies that $u \equiv 0$.

Proof of Theorem 1.3. Let α and β be the same as in (15). Then the assertion of the representation theorem is expressed by the equation

$$\alpha - i\beta = 0.$$

Now multiply (11) by $\frac{\partial}{\partial R}\left(\frac{\sin kR}{R}\right)$ and (12) by $-\frac{\partial}{\partial R}\left(\frac{\cos kR}{R}\right)$ and add. After some simplification we get

$$\frac{\partial}{\partial R}\left(\frac{\sin kR}{R}\right)\alpha + \frac{\partial}{\partial R}\left(\frac{\cos kR}{R}\right)\beta = \frac{k}{4\pi R^2}\iint_{\Sigma_R} \frac{\partial u}{\partial R}\, dS. \tag{1.25}$$

Now multiply (25) by $-ik$ and add this result to (15). We obtain

$$\alpha\left[\frac{\partial}{\partial R}\left(\frac{\sin kR}{R}\right) - ik\frac{\sin kR}{R}\right] + \beta\left[\frac{\partial}{\partial R}\left(\frac{\cos kR}{R}\right) - ik\frac{\cos kR}{R}\right] = \frac{k}{4\pi R^2}\iint_{\Sigma_R}\left(\frac{\partial u}{\partial R} - iku\right)dS. \tag{1.26}$$

Applying Schwarz' inequality to the right member of (26) yields

$$\left|(\alpha - i\beta)\frac{k\cos kR}{R} + (i\alpha+\beta)\left(-\frac{k\sin kR}{R}\right) - \alpha\frac{\sin kR}{R^2} - \beta\frac{\cos kR}{R^2}\right| \leq \frac{k}{2\sqrt{\pi}R}\sqrt{\iint_{\Sigma_R}\left|\frac{\partial u}{\partial R} - iku\right|^2 dS}\ . \tag{1.27}$$

The radiation condition implies that the right member of (27) vanishes faster than $1/R$ as R tends to infinity. α and β are independent of R. The coefficient of $\alpha - i\beta$ in the left member of (27) is exactly $O(1/R)$. The remaining terms in the left member of (27) are $O(1/R^2)$. Thus since the right member tends to zero faster than $1/R$ as R tends to infinity, we must have

$\alpha - i\beta = 0$. This proves the theorem.

The proof of the corollary of Theorem 3 is similar to the proof of the Theorem itself and will not be given.

Proof of Theorem 1.4. The proof proceeds exactly as the proof of Theorem 3. The assertion of Theorem 4 is expressed by the equation $\alpha - i\beta = 0$. We obtain in place of (27) the equation

$$\left|(\alpha_R - i\beta_R)\frac{k\cos kR}{R} + (i\alpha_R + \beta_R)\left(-\frac{k\sin kR}{R}\right) - \alpha_R\frac{\sin kR}{R^2} - \beta_R\frac{\cos kR}{R^2}\right| \leq \frac{k\sqrt{A(R)}}{4\pi R^2}\sqrt{\iint_{E_R}\left|\frac{\partial u}{\partial R} - iku\right|^2 dS}\,. \tag{1.28}$$

Here

$$\alpha_R = u(P) - \frac{1}{4\pi}\iint_{B_R}\left(\frac{\cos k|P-Q|}{|P-Q|}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\frac{\cos k|P-Q|}{|P-Q|}\right) dS \tag{1.29}$$

and

$$\beta_R = \frac{1}{4\pi}\iint_{B_R}\left(\frac{\sin k|P-Q|}{|P-Q|}\frac{\partial u}{\partial n} - u\frac{\partial}{\partial n}\frac{\sin k|P-Q|}{|P-Q|}\right) dS. \tag{1.30}$$

Since $A(R) \leq O(R^2)$, the radiation condition (5) implies that the right member of (28) tends to zero faster than $1/R$. By hypothesis the following limits exist:

$$\lim_{R\to\infty}\alpha_R = \alpha \qquad \lim_{R\to\infty}\beta_R = \beta$$

$$\lim_{R\to\infty}(\alpha_R - i\beta_R) = -i\lim_{R\to\infty}(i\alpha_R + \beta_R) = \alpha - i\beta. \tag{1.31}$$

Thus by arguments similar to those used in the proof of Theorem 3, (28) implies that $\alpha - i\beta = 0$ and this proves the theorem.

The proof of the corollary is similar to the proof of Theorem 4 and will be omitted.

Proof of Theorem 1.5. The proof proceeds as the proof of Theorem 1. We obtain in place of (16) the equation

$$\left|\alpha_R\frac{\sin kR}{R} + \beta_R\frac{\cos kR}{R}\right| \leq \frac{k\sqrt{A(R)}}{2\sqrt{\pi}\,R}\sqrt{\iint_{E_R}|u|^2 dS}\,. \tag{1.32}$$

Here, because u vanishes on B, we have

$$\alpha_R = u(P) - \frac{1}{4\pi}\int_{B_R}\!\!\int \frac{\cos k|P-Q|}{|P-Q|}\,\frac{\partial u}{\partial n}\,dS \tag{1.33}$$

and

$$\beta_R = \frac{1}{4\pi}\int_{B_R}\!\!\int \frac{\sin k|P-Q|}{|P-Q|}\,\frac{\partial u}{\partial n}\,dS. \tag{1.34}$$

By hypothesis the following limits exist:

$$\lim_{R\to\infty} \alpha_R = \alpha \qquad \lim_{R\to\infty} \beta_R = \beta.$$

Thus the left member of (32) is exactly $O(1/R)$ unless $\alpha = 0$ and $\beta = 0$. Now, exactly as in the proof of Lemma 1, the radiation condition together with the fact that u vanishes on B implies that $\int_{E_R}\!\!\int |u|^2 dS$ tends to zero as R tends to infinity. Thus the right member of (32) tends to zero faster than 1/R. Combining these facts we see by arguments already used that (32) implies that α and β are zero. From (32), which defines α_R, we see that $\alpha = 0$ is equivalent to (8). Similarly (34) and $\beta = 0$ imply (9). This proves the theorem.

2. The reduced wave equation in a medium with a variable index of refraction

In this section we discuss solutions u of the equation

$$\Delta u + h^2(x)u = 0. \tag{2.1}$$

Here $x = (x_1, x_2, x_3)$. In particular we consider the case in which

$$\lim_{|x|\to\infty} h^2(x) = k^2 = \text{constant}. \tag{2.2}$$

We show that the Rellich growth estimate (Theorem 1.1) is valid when u is defined and bounded in all of space. From this result we deduce two statements concerning the non-existence of eigenvalues. The first is that for no positive value of k^2 does there exist a solution v of $\Delta v + k^2 v = 0$ which is bounded and quadratically integrable over all of space. The second is the same statement for the equation $\Delta v + W(x)v = -k^2 v$ for a suitable class of $W(x)$. We then prove that there exists no outgoing solution of (1) which is defined and bounded in all of space.

When u is defined in the complement of a bounded domain, we show that there exists a solution v of $\triangle v + k^2 v = 0$ such that v approaches u as $|x|$ tends to infinity. The rate of this approach is comparable to the rate with which $h^2(x)$ approaches k^2 at infinity. We call this result the osculation theorem. From this theorem we deduce that the Rellich growth estimate is valid for u when it is defined and bounded in the complement of a bounded domain. This last result leads to a uniqueness theorem for the exterior boundary-value problem for u.

The osculation theorem is similar to a theorem proved by L.Bers [4]. Bers considers the approximation in the neighborhood of a finite point of a solution of a linear elliptic partial differential equation by homogeneous harmonic polynomials. The solution has a zero of a certain order at this point and the degree of the approximating polynomial is related to this order.

We now state our results; the proofs are given immediately afterwards.

Let $N > 2$ and let

$$(2.3) \qquad |h^2(x)-k^2| \leq O(|x|^{-N}).$$

Let dS denote a surface element and dV a volume element. The radiation condition is

$$(2.4) \qquad \lim_{|x|\to\infty} \int\int_{\Sigma_R} \left| \frac{\partial u}{\partial r} - ih(x)u \right|^2 dS = \lim_{R\to\infty} \int\int_{\Sigma_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 dS = 0.$$

Σ_R is the surface of a sphere of radius R with some fixed point P as center.

Theorem 2.1. Let u be a solution of (1) which is defined and bounded in all of space, V. Let

$$(2.5) \qquad \frac{1}{4\pi} \int\int\int_V \frac{|k^2-h^2(x)|}{|x|} dV < 1.$$

Then

$$(2.6) \qquad \lim_{R\to\infty} \int\int_{\Sigma_R} |u|^2 dS \neq 0$$

unless $u \equiv 0$.

Corollary 1. There exists no positive value of k^2 for which $\triangle u + k^2 u = 0$ possesses a bounded solution which is quadratically integrable over all of space.

Corollary 2. If

$$\frac{1}{4\pi} \int\int\int_{-\infty}^{\infty} \left| \frac{W(x)}{x} \right| dV < 1,$$

then there exists no positive value of k^2 for which $\triangle u + W(x)u = -k^2u$ has a bounded solution which is quadratically integrable over all of space.

Theorem 2.2. Under the hypothesis of Theorem 1, the only solution u of (1) which satisfies the radiation condition (4) is the trivial solution $u \equiv 0$.

For the following results u is defined and bounded in the exterior D of a bounded domain with boundary B. B is such that Green's theorem may be applied to the domain $D_R = D \cap (\text{interior of } \sum_R)$, for $\sum_R$ containing B.

Lemma 2.1. Let u satisfy (1) and the radiation condition (4). Then u is a solution of the integral equation

$$u(P) = \frac{1}{4\pi} \iint_B \left[\frac{e^{ik|P-Q|}}{|P-Q|} \frac{\partial u}{\partial n} - u \frac{\partial}{\partial n} \left(\frac{e^{ik|P-Q|}}{|P-Q|} \right) \right] dS$$
$$- \frac{1}{4\pi} \iiint_D u \frac{e^{ik|P-Q|}}{|P-Q|} (k^2 - h^2(Q)) dV.$$

Lemma 2.2. Let u satisfy (1) and the radiation condition (4). Let u or the values of the normal derivative of u on B vanish. Then

$$\lim_{R \to \infty} \iint_{\sum_R} \left(h^2(x)|u|^2 + \left| \frac{\partial u}{\partial R} \right|^2 \right) dS = 0. \tag{2.7}$$

Theorem 2.3 (osculation theorem). Let u be a solution of (1) and satisfy the radiation condition (4). If $k^2 - h^2(x) = O\left(\frac{1}{|x|^N}\right)$, there exists a solution v of $\triangle v + k^2 v = 0$ such that $u - v = O(|x|^{-N+2+\varepsilon})$. ε is an arbitrarily small positive number. Furthermore, $v \not\equiv 0$ if $u \not\equiv 0$.

Remark: A similar theorem is valid when the point at infinity is replaced by a finite point.

Corollary 1. If we assume the hypothesis of Theorem 3 and if $N > 3$, we have

$$\lim_{R \to \infty} \iint_{\sum_R} |u|^2 \, dS \neq 0 \tag{2.8}$$

unless $u \equiv 0$.

Corollary 2 (uniqueness theorem). There exists at most one solution u of (1) which also satisfies (4) with prescribed boundary values or prescribed values of its normal derivative along B.

Now we give the proofs of the theorems of this section.

Proof of Theorem 2.1. Let V_R be the interior of $\sum_R$. Let u be a solution of (1) and let v be a solution of $\triangle v + k^2 v = 0$. Since $v \triangle u - u \triangle v = uv(k^2-h^2(x))$, we have by Green's theorem

$$\iiint_{V_R} uv(k^2-h^2(x))dV = \iint_{\sum_R} (v \frac{\partial u}{\partial R} - u \frac{\partial v}{\partial R})\, dS. \tag{2.9}$$

Now we use the same method by which we obtained (1.14); from (9) we get

$$\begin{aligned} &\frac{\sin kR}{R}\left[u(P) + \frac{1}{4\pi}\iiint_{V_R} u \frac{\cos k|P-Q|}{|P-Q|}(k^2-h^2(Q))dV\right] \\ &- \frac{\cos kR}{4\pi R}\iiint_{V_R} u \frac{\sin k|P-Q|}{|P-Q|}(k^2-h^2(Q))dV = \frac{k}{4\pi R^2}\iint_{\sum_R} u\, dS. \end{aligned} \tag{2.10}$$

We introduce the abbreviations α_R and β_R for the coefficients of $\frac{\sin kR}{R}$ and $-\frac{\cos kR}{R}$ respectively in (10). Then applying Schwarz' inequality to the right member of (10), we obtain

$$\left|\alpha_R \frac{\sin kR}{R} - \beta_R \frac{\cos kR}{R}\right| \leq \frac{k}{2\sqrt{\pi} R}\sqrt{\iint_{\sum_R} |u|^2\, dS}\,. \tag{2.11}$$

Since u is bounded and $|k^2-h^2(x)| \leq O\left(\frac{1}{|x|^N}\right)$, $N > 2$, the limits as R tends to infinity of the volume integrals in (10) exist. Then let

$$\alpha = \lim_{R\to\infty} \alpha_R \quad \text{and} \quad \beta = \lim_{R\to\infty} \beta_R \;.$$

Now if, contrary to the theorem, $\lim_{R\to\infty} \iint_{\sum_R} |u|^2\, dS = 0$, then the right member of (11) tends to zero faster than $1/R$ as $R \to \infty$. In this event (11) implies that $\alpha = \beta = 0$ for all choices of P. Now by definition of α_R, $\alpha = 0$ is equivalent to

$$u(P) = -\frac{1}{4\pi}\iiint_V \frac{\cos k|P-Q|}{|P-Q|}(k^2-h^2(Q))dV. \tag{2.12}$$

From (12) we have

$$\operatorname{lub} |u(P)| \leq \operatorname{lub} |u(P)| \left| \frac{1}{4\pi} \iiint_V \frac{|k^2-h^2(Q)|}{|P-Q|} \, dV \right| .$$

If $u \not\equiv 0$ this inequality and the hypothesis imply that

$$\operatorname{lub} |u(P)| < \operatorname{lub} |u(P)|.$$

This absurdity implies the correctness of the theorem.

Corollaries 1 and 2 of Theorem 1 both follow immediately from the theorem.

Proof of Theorem 2.2. Using the same method by which (1.26) was obtained, we get

$$\left[k \frac{\cos kR}{R} - \frac{\sin kR}{R^2} - ik \frac{\sin kR}{R}\right]\alpha_R - \left[-k \frac{\sin kR}{R} - \frac{\cos kR}{R^2} - ik \frac{\cos kR}{R}\right] \beta_R$$

(2.13)

$$= \frac{k}{4\pi R^2} \iint_{\sum_R} \left(\frac{\partial u}{\partial R} - ih(Q)u \right) dS.$$

Here

$$\alpha_R = u(P) + \frac{1}{4\pi} \iiint_{V_R} u \frac{\cos k|P-Q|}{|P-Q|} (k^2-h^2(Q))dV$$

and

$$\beta_R = \frac{1}{4\pi} \iiint_{V_R} u \frac{\sin k|P-Q|}{|P-Q|} (k^2-h^2(Q)) \, dV.$$

V_R is the interior of $\sum_R$. We rewrite (13) as

$$k(\alpha_R+i\beta_R) \frac{\cos kR}{R} - k(i\alpha_R-\beta_R) \frac{\sin kR}{R} - \alpha_R \frac{\sin kR}{R^2} + \beta_R \frac{\cos kR}{R^2}$$

(2.14)

$$= \frac{k}{4\pi R^2} \iint_{\sum_R} \left(\frac{\partial u}{\partial R} - ih(Q)u \right) dS.$$

Applying Schwarz' inequality to the right member of (14) we get

$$\left| (\alpha_R+i\beta_R)k \frac{\cos kR}{R} - i(\alpha_R+i\beta_R)k \frac{\sin kR}{R} - \alpha_R \frac{\sin kR}{R^2} + \beta_R \frac{\cos kR}{R^2} \right|$$

(2.15)

$$\leq \frac{k}{2\sqrt{\pi}R} \sqrt{\iint_{\sum_R} \left| \frac{\partial u}{\partial R} - ih(Q)u \right|^2 dS} .$$

u is bounded by hypothesis. Thus since $|k^2-h^2(x)| \leq O(\frac{1}{|x|} N)$, $N > 2$, the limits as R tends to infinity of α_R and β_R exist. Call these limits α and β respectively. The radiation condition (4) implies that the right member of (15) tends to zero faster than $1/R$ as $R \to \infty$. The coefficient of $\alpha_R + i\beta_R$ in the left member of (15) is exactly $O(1/R)$ while the remaining terms in the left member are $O(1/R^2)$. Thus (15) implies that

$$\lim_{R\to\infty} \alpha_R + i\beta_R = \alpha + i\beta = 0. \tag{2.16}$$

From the definition of α_R and β_R, $\alpha+i\beta = 0$ is equivalent to

$$u(P) = -\frac{1}{4\pi}\iiint u \frac{e^{ik|P-Q|}}{|P-Q|}(k^2-h^2(Q))dV. \tag{2.17}$$

Here integration is over all of space. From (17) we have

$$\text{lub } |u(P)| \leq \text{lub } |u(P)| \left| \frac{1}{4\pi}\iiint \frac{|k^2-h^2(Q)|}{|P-Q|} dV \right| . \tag{2.18}$$

Therefore if $u \neq 0$, it follows from the hypothesis that

$$\text{lub } |u(P)| < \text{lub } |u(P)|.$$

This absurdity implies that $u = 0$ and proves the theorem.

Proof of Lemma 2.1. Since $\triangle u + k^2u = (k^2-h^2(x))u$, Lemma 1 is essentially the assertion of the corollary to Theorem 1.3 with $f = (k^2-h^2(x))u$. The integral,

$$\iiint_V u \frac{e^{ik|P-Q|}}{|P-Q|}(k^2-h^2(Q))dV,$$

exists since u is bounded and $|k^2-h^2(Q)| < O(\frac{1}{|x|^N})$, $N > 2$. The proof of Lemma 1 is omitted since it is similar to the proof of the corollary to Theorem 1.3.

The proof of Lemma 2 is similar to the proof of Lemma 1.1 and is omitted.

Proof of Theorem 2.3 (osculation theorem). From the integral equation (6) we have, denoting the boundary integrals by $v_0(x)$, P by x, and Q by ξ,

$$u = v_0 - \frac{1}{4\pi}\iiint_D u(\xi)\frac{e^{ik|x-\xi|}}{|x-\xi|}(k^2-h^2(\xi))dV. \tag{2.19}$$

v_0 is a solution of $\triangle v + k^2v = 0$ since it is a sum of single and double layer distributions.

Let

$$f(|\xi|) = \frac{e^{ik|x-\xi|}}{|x-\xi|} = \sum_{n=0}^{\infty} a_n|\xi|^n \tag{2.20}$$

and

$$g(|\xi|) = \frac{1}{|x-\xi|} = \sum_{n=0}^{\infty} b_n |\xi|^n = \sum_{\nu=0}^{\infty} \frac{|\xi|^n P_n(\gamma)}{|x|^{n+1}} . \tag{2.21}$$

Here $P_n(\gamma)$ is the n-th Legendre polynomial of the cosine of the angle between x and ξ. These Taylor series converge uniformly and absolutely for $|\xi| < |x|$.

Now rewrite (19) as

$$u = v_0 - \frac{1}{4\pi} \iiint_D u(\xi)(k^2-h^2(\xi)) \left[\sum_{n=0}^{A} a_n |\xi|^n + \frac{e^{ik|x-\xi|}}{|x-\xi|} - \sum_{n=0}^{A} a_n |\xi|^n \right] d\xi \tag{2.22}$$

or

$$u = v_0 + v_1 - \frac{1}{4\pi} \iiint_D u(\xi)(k^2-h^2(\xi)) \left[\frac{e^{ik|x-\xi|}}{|x-\xi|} - \sum_{n=0}^{A} a_n |\xi|^n \right] d\xi . \tag{2.23}$$

From (20) we see that each a_n is a linear combination of derivatives with respect to $\xi(\xi_1,\xi_2,\xi_3)$ of the fundamental solution of the reduced wave equation $e^{ik|x-\xi|}/|x-\xi|$, evaluated at $\xi = 0$. Since $|x| > 0$ each a_n is a solution of the reduced wave equation and thus v_1 is also a solution.

Since $k^2-h^2(\xi) = O(|\xi|^{-N})$, and since u is bounded, the volume integral which appears explicitly in (23) converges absolutely if $A < N-3$. Now replace D by the exterior of a sphere whose interior contains B and which is so large that this volume integral is less than $\text{lub}|u|/2$ in magnitude. If the v_0+v_1 corresponding to this sphere were zero then from (23), we would get $\text{lub}|u| < \text{lub}|u|/2$. This absurdity implies that u is zero in this even. Thus we may assume without loss of generality that D is the exterior of a sphere of radius R, and we have $v_0+v_1 \neq 0$ if $u \not\equiv 0$.

We must now estimate the volume integral which appears explicitly in (23). Since the series (20) converges absolutely for $|\xi| < |x|$, we may rewrite (23) as*

$$\begin{aligned} u = v_0+v_1 &- \frac{1}{4\pi} \iiint_{R<|\xi|<\theta|x|} u(\xi)(k^2-h^2(\xi)) \sum_{n=A+1}^{\infty} a_n |\xi|^n d\xi \\ &- \frac{1}{4\pi} \iiint_{|\xi|\geq\theta|x|} u(\xi)(k^2-h^2(\xi)) \left[\frac{e^{ik|x-\xi|}}{|x-\xi|} - \sum_{n=0}^{A} a_n |\xi|^n \right] d\xi, \end{aligned} \tag{2.24}$$

or

$$u = v_0 + v_1 + I_1 + I_2. \tag{2.25}$$

Now if we regard $|\xi|$ as a complex variable, we have

$$|a_n| = \left| \frac{1}{2\pi i} \int_C \frac{f(|\xi|)}{|\xi|^{n+1}} d|\xi| \right| . \tag{2.26}$$

---

* The technique of splitting the volume integral into the sum $I_1 + I_2$ in (24) (25) is employed by L. Bers[4].

C is a circle in the complex $|\xi|$-plane with center at $|\xi| = 0$ and with radius $r < |x|$. We will show that

$$(2.27) \qquad |a_n| \leq \frac{1}{2\pi} \int_C \frac{|f(|\xi|)|\,|d|\xi||}{\left|\,|\xi|^{n+1}\right|} \leq \frac{1}{2\pi}\frac{1}{r^n} \int_0^{2\pi} |g(re^{i\theta})|\,d\theta .$$

This may be seen as follows. First,

$$(2.28) \qquad \begin{aligned} \left|f(|\xi|)\right| &= \left|g(|\xi|)\right| \left|e^{ik|x-\xi|}\right| \\ &= \left|g(|\xi|)\right| \left|\exp\left[ik\sqrt{|\xi|^2 + |x|^2 - 2|\xi||x|\gamma}\,\right]\right| . \end{aligned}$$

Now $(|\xi|^2 + |x|^2 - 2|\xi||x|\gamma) = \triangle$ is a single-valued analytic function of the complex variable $|\xi|$. Since the amplitude of $\triangle$ for $|\xi|$ real is 0, we may consider the amplitude of $\triangle$ to be between 0 and 2π for all complex $|\xi|$. Since $\triangle \neq 0$ for $||\xi|| \leq r < |x|$, the amplitude of $\sqrt{\triangle}$ is between 0 and π. Thus

$$\left|e^{ik\sqrt{\triangle}}\right| \leq 1 \quad \text{and} \quad \left|f(|\xi|)\right| \leq \left|g(|\xi|)\right| \qquad (k > 0).$$

Now (27) implies

$$(2.29) \qquad |a_n| \leq \frac{1}{r^n} |b_o| .$$

From (21) we find that $|b_o| = \frac{1}{|x|}$. If we let $r = \lambda|x|$, $\lambda < 1$, (29) becomes

$$(2.30) \qquad |a_n| < \frac{\lambda}{|\lambda x|^{n+1}} .$$

Let $M = \text{lub}|u(\xi)|$ and let $|k^2 - h^2(\xi)| \leq K|\xi|^{-N}$. Then from (24) and (25), we have

$$(2.31) \qquad |I_1| \leq \lambda MK \int_R^{\theta|x|} |\xi|^{-N} \sum_{A+1}^{\infty} \frac{|\xi|^n}{|\lambda x|^{n+1}} |\xi|^2 \, d|\xi| .$$

Let $A = N-4$ and make $\theta < \lambda$. Then

$$(2.32) \qquad \begin{aligned} |I_1| &\leq \lambda MK \sum_{N-3}^{\infty} \frac{1}{|\lambda x|^{n+1}} \int_R^{\theta|x|} |\xi|^{-N+n+2} \, d|\xi| \\ &= \lambda MK \left\{ \frac{\log \theta|x|/R}{|\lambda x|^{N-2}} + \sum_{N-2}^{\infty} \left[\frac{(\theta|x|)^{-N+n+3}}{(\lambda|x|)^{n+1}(-N+m+3)} - \frac{R^{-N+n+3}}{|\lambda x|^{n+1}(-N+n+3)} \right] \right\} \\ &\leq \lambda MK \left\{ \frac{\log \theta|x|/R}{|\lambda x|^{N-2}} + \frac{\theta|x|}{|\lambda x|^{N-1}} \sum_{0}^{\infty} \left(\frac{\theta|x|}{\lambda|x|}\right)^n + \frac{R}{|\lambda x|^{N-1}} \sum_{0}^{\infty} \frac{R^n}{|\lambda x|^n} \right\} \\ &= \lambda MK \left\{ \frac{\log \theta|x|/R}{|\lambda x|^{N-2}} + \frac{\theta}{\lambda^{N-1}} \frac{1}{|x|^{N-2}} \frac{1}{1-\frac{\theta}{\lambda}} + \frac{R}{\lambda^{N-1}|x|^{N-1}} \frac{1}{1-\frac{R}{\lambda|x|}} \right\} . \end{aligned}$$

Therefore

(2.33) $$|I_1| \leq O(|x|^{-N+2+\varepsilon}).$$

We note that ε is an arbitrarily small positive number since it arises to account for the term $\log \theta|x|/R$ in (32).

Now

$$|I_2| \leq \frac{MK}{4\pi} \iiint\limits_{\theta|x|<|\xi|} |\xi|^{-N} \left[\frac{1}{|x-\xi|} + \lambda \sum_{n=0}^{N-4} \frac{|\xi|^n}{|\lambda x|^{n+1}} \right] d\xi$$

(2.34) $$\leq MK \left[\frac{1}{4\pi(\theta|x|)^{N-2-\varepsilon}} \iiint\limits_{\theta|x|<|\xi|} \frac{d|\xi|}{|\xi|^{\varepsilon}|x-\xi|} + \lambda \sum_{n=0}^{N-4} \frac{1}{\lambda|x|^{n+1}} \int_{\theta|x|}^{\infty} |\xi|^{-N+n+2}\, d|\xi| \right]$$

$$\leq MK \left[O(|x|^{-N+2+\varepsilon}) + \lambda \sum_{n=0}^{N-4} \frac{(\theta|x|)^{-N-n+3}}{(\lambda|x|)^{n+1}|-N+n+3|} \right].$$

Here ε is an arbitrarily small positive number. Therefore

(2.35) $$|I_2| \leq O(|x|^{-N+2+\varepsilon}).$$

Insert (33) and (35) into (23). We obtain

(2.36) $$u = v_0 + v_1 + O(|x|^{-N+2+\varepsilon}).$$

and the theorem is proved.

Proof of Corollary 1. From the osculation theorem we have

(2.37) $$u = v + I$$

where $\triangle v + k^2 v = 0$ and

(2.38) $$|I| \leq O(|x|^{-N+2+\varepsilon}).$$

We know also from this theorem that $v \neq 0$ if $u \neq 0$. Suppose then that $u \neq 0$. If $\bar{v}$ and $\bar{I}$ are the complex conjugates of v and I, we have

$$\iint_{\Sigma_R} |u|^2 dS = \iint_{\Sigma_R} |v+I|^2 dS$$

(2.39) $$= \iint_{\Sigma_R} |v|^2 dS + \iint_{\Sigma_R} (\bar{v}\, I + v\, \bar{I}) dS + \iint_{\Sigma_R} |I|^2 dS.$$

Since $v \not\equiv 0$, Theorem 1.1 (Rellich's estimate) asserts that

$$\lim_{R\to\infty} \iint_{\Sigma_R} |v|^2 \, dS \neq 0. \tag{2.40}$$

Since $N > 3$ and ε is an arbitrarily small positive number, there exists $\delta > 0$ such that $|I| \leq O(|x|^{-1-\delta})$. Thus as R tends to infinity the magnitude of the last two integrals in the right member of (39) is of a lower order than

$$\iint_{\Sigma_R} |v|^2 \, dS.$$

Thus (39) implies

$$\lim_{R\to\infty} \iint_{\Sigma_R} |u|^2 \, dS \neq 0 \tag{2.41}$$

and this proves the corollary.

Proof of Corollary 2 (uniqueness theorem). Suppose, in contradiction to the corollary, that v_1 and v_2 are two solutions of (1) which satisfy the radiation condition, and have the same boundary values or the same normal derivatives along B. Let $v = v_1 - v_2$. An application of Schwarz' inequality shows that v satisfies the radiation condition. Since the values of v along B or the values of the normal derivative of v along B vanish, Lemma 2 implies that

$$\lim_{R\to\infty} \iint_{\Sigma_R} |v|^2 \, dS = 0.$$

But this implies $v \equiv 0$ by Corollary 1.

3. The L^2-maximum principle

In this section we will consider solutions of the reduced wave equation $\Delta u + k^2 u = 0$. u is defined in the exterior of a bounded domain with boundary B and satisfies the radiation condition. We will obtain an estimate for the L^2-norm of the normal derivative of u along B in terms of the boundary values of u. From this estimate and the representation theorem for u, it follows that the magnitude of u is bounded by a homogeneous quadratic functional of its boundary values. We call this result the L^2-maximum principle, because of its similarity to the usual maximum principle. This principle is of interest because it applies to a boundary-value problem for which the ordinary maximum principle is not true.

Our estimate is similar to one obtained in 1951 by Vishik[19] for any solution u of Laplace's equation defined within a sphere of radius R. Vishik showed that

$$\iint_{r=R} \left(\frac{\partial u}{\partial r}\right)^2 R^2 \sin \phi \, d\phi \, d\theta \leq \iint_{r=R} \left[\left(\frac{\partial u}{\partial \theta}\right)^2 + \frac{1}{\sin^2 \theta} \left(\frac{\partial u}{\partial \phi}\right)^2\right] R^2 \sin \phi \, d\phi d\theta .$$

He remarked that similar inequalities are valid for more general elliptic equations and for more general bounded domains. In 1956, L. Hörmander[5] deduced estimates of this type for solutions of a class of second-order elliptic equations. Hörmander requires that these solutions be defined in a compact domain.

The estimates in this chapter are new in two respects. First and most important, they are concerned with an unbounded domain. Second, they are concerned with a complex-valued solution of the reduced wave equation.

In the following paragraphs, we state our results. Then we furnish proofs.

Let u be a solution of the reduced wave equation, $\Delta u + k^2 u = 0$, in the exterior D of a bounded domain with boundary B. Let u satisfy the n-dimensional radiation condition

$$\lim_{R \to \infty} \iint_{\Sigma_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 R^{n-1} \, d\omega = 0. \tag{3.1}$$

Here Σ_R is the surface of an n-dimensional sphere of radius R with some fixed point in n-space as center. A double integral indicates a surface integration and $R^{n-1} \, d\omega$ denotes a surface element. We will prove the following lemmas:

Lemma 3.1. Let B be the surface of a sphere Σ of radius a, and let $2a \operatorname{Im} k \geq n-1$. Then if $1-2\varepsilon^2 > 0$,

$$\int_{\Sigma_R}\int \left| \frac{\partial u}{\partial r} \right|^2 ds \le \frac{1}{1-2\varepsilon^2} \left[\int_{\Sigma}\int |\nabla_t u|^2 a^{n-1} d\omega \right.$$

(3.2)

$$\left. + \frac{|k|^2}{\varepsilon^2}\left(1 + \frac{|k|^2}{2(\mathrm{Im} k)^2}\right) \int_{\Sigma}\int |u|^2 a^{n-1} d\omega \right] .$$

$|\nabla_t u|^2$ denotes the square of the magnitude of the tangential component of the gradient of u. This is, in two, three and n dimensions respectively,

$$|\nabla_t u|^2 = r^{-2} \left| \frac{\partial u}{\partial \theta} \right|^2$$

$$|\nabla_t u|^2 = r^{-2} \left(\left| \frac{\partial u}{\partial \theta} \right|^2 + \frac{1}{\sin^2\theta} \left| \frac{\partial u}{\partial \phi} \right|^2 \right)$$

$$|\nabla_t u|^2 = r^{-2} \left[(\sin\theta_1,\ldots,\sin\theta_{n-2})^{-2} \left| \frac{\partial u}{\partial \phi} \right|^2 + \left| \frac{\partial u}{\partial \theta_1} \right|^2 + (\sin\theta_1)^{-2} \left| \frac{\partial u}{\partial \theta_2} \right|^2 \right.$$

$$\left. + \cdots + (\sin\theta_1,\ldots,\sin\theta_{n-3})^{-2} \left| \frac{\partial u}{\partial \theta_{n-2}} \right|^2 \right] .$$

Lemma 3.2. Let B be a simple closed curve. Let the transformation $x = x(\alpha,\beta)$, $y = y(\alpha,\beta)$ define an orthogonal coordinate system in D. The functions $x(\alpha,\beta)$ and $y(\alpha,\beta)$ are twice continuously differentiable with respect to α and β. Let B be a coordinate curve corresponding to some constant value of α. Let α be arclength along the curves β = constant. Let

$$F^{-2} = \left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2$$

and

$$G^{-2} = \left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2 .$$

Let ds denote a line element. If $1-2\varepsilon^2 > 0$ and

$$\mathrm{Im}\, k \ge \left| G^{-1} \frac{\partial G}{\partial \alpha} \right| \tag{3.3}$$

then

$$\int_B \left| \frac{\partial u}{\partial n} \right|^2 ds \le \frac{1}{1-2\varepsilon^2} \left[\int_B \left| \frac{\partial u}{\partial \beta} \right|^2 ds + \frac{|k|^2}{\varepsilon^2}\left(1 + \frac{|k|^2}{2(\mathrm{Im} k)^2}\right) \int_B |u|^2 ds \right] . \tag{3.4}$$

A similar result is valid for higher dimensions.

We will see below that when B is convex the coordinate curves may be chosen to be the normals and parallels to B. If B corresponds to $\alpha = 0$ and

$\kappa = \kappa(\beta)$ is the curvature of B, then the condition (3) becomes $\operatorname{Im} k \geq \left|\frac{\kappa}{1+a\kappa}\right|$.

Lemma 3.3. Let B be the surface of a sphere $\sum$ of radius a. If $1-\varepsilon^2 > 0$, k real, and ka is sufficiently large, then

$$(3.5) \qquad \iint_{\Sigma} \left|\frac{\partial u}{\partial r}\right|^2 a^{n-2} d\omega \leq \frac{1}{1-\varepsilon^2}\left[\frac{|k|^2}{\varepsilon^2}\iint_{\Sigma} |u|^2 a^{n-2} d\omega + \iint_{\Sigma} |\nabla_t u|^2 a^{n-2} d\omega\right].$$

Each of these lemmas implies a particular L^2-maximum principle. In particular, Lemma 3 implies

Theorem 3.1 (L^2-maximum principle). Let u be a solution of $\Delta u + k^2 u = 0$ in the exterior D of a sphere of radius a. Let $\sum$ denote the surface of this sphere and let u satisfy the radiation condition (1). If $1-\varepsilon^2 > 0$, k real, and ka is sufficiently large, then

$$(3.6) \qquad \begin{aligned} |u(P)| &\leq a\left(\iint_{\Sigma} |u|^2 a^{n-2} d\omega\right)^{1/2} \\ &\quad + b\left(\frac{1}{1-\varepsilon^2}\left[\frac{|k|^2}{\varepsilon^2}\iint_{\Sigma} |u|^2 a^{n-2} d\omega + \iint_{\Sigma} |\nabla_t u|^2 a^{n-2} d\omega\right]\right)^{1/2} \end{aligned}$$

where

$$(3.7) \qquad a = \left(\iint_{\Sigma} \left|\frac{\partial}{\partial r} F(k|P-Q|)\right|^2 a^{n-2} d\omega\right)^{1/2}$$

and

$$(3.8) \qquad b = \left(\iint_{\Sigma} |F(k|P-Q|)|^2 a^{n-2} d\omega\right)^{1/2}.$$

$F(k|P-Q|)$ is the fundamental solution of $\Delta u + k^2 u = 0$ which satisfies the radiation condition. In two, three, and n dimensions we have respectively

$$(3.9) \qquad \begin{aligned} F(k|P-Q|) &= -\frac{i}{4} H_0^{(1)}(k|P-Q|) \\ F(k|P-Q|) &= \frac{1}{4\pi}\frac{e^{ik|P-Q|}}{|P-Q|} \\ F(k|P-Q|) &= \Gamma\left(\frac{n}{2}\right) H^{(1)}_{\frac{n-2}{2}}(kr)\left(2\pi^{n/2} r^{(n-2)/2}\right)^{-1}. \end{aligned}$$

We now give the proofs for all these results.

We first observe that Theorem 1 follows from Lemma 3 and the representation theorem, namely

$$(3.10) \qquad u(P) = \iint_{\Sigma} \left(u \frac{\partial}{\partial r} F(k|P-Q|) - \frac{\partial u}{\partial r} F(k|P-Q|) \right) a^{n-2} d\omega \; .$$

This representation is valid for outgoing solutions of the reduced wave equation (see Theorem 1.3). Application of Schwarz' inequality to (10) and use of Lemma 3 implies Theorem 1.

The proofs of the lemmas will be conducted in two dimensions in order to attain simplicity in our arguments. In what follows, C denotes a circle, ds a line element, and dS a surface element.

Proof of Lemma 3.1. The proof proceeds by extending the technique of Vishik[19]. Let D_R denote the annulus between C and a larger concentric circle C_R of radius R, and let a bar over a quantity denote the complex conjugate of that quantity. In two-dimensional polar coordinates we have

$$(3.11) \qquad \Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \; .$$

Now multiply the reduced wave equation by $\frac{\partial \bar{u}}{\partial r}$ and integrate the result over D_R. We obtain

$$(3.12) \qquad \iint_{D_R} \Delta u \frac{\partial \bar{u}}{\partial r} r \, dr \, d\theta + k^2 \iint_{D_R} u \frac{\partial \bar{u}}{\partial r} r \, dr \, d\theta = 0.$$

Inserting (11) into (12) and integrating the result by parts with respect to r, we obtain

$$\int_{C_R} \left| \frac{\partial u}{\partial r} \right|^2 R \, d\theta - \int_{C} \left| \frac{\partial u}{\partial r} \right|^2 r \, d\theta - \iint_{D_R} \frac{\partial u}{\partial r} \frac{\partial^2 \bar{u}}{\partial r^2} r \, dr \, d\theta$$

$$(3.13) \qquad + \iint_{D_R} \frac{\partial \bar{u}}{\partial r} \left[\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + k^2 u \right] r \, dr \, d\theta = 0.$$

From (11) we see that the reduced wave equation contains the term $\frac{\partial^2 u}{\partial r^2}$. Solving the reduced wave equation for $\frac{\partial^2 u}{\partial r^2}$, and inserting the result in (13), we get

$$\int_{C_R} \left|\frac{\partial u}{\partial r}\right|^2 R d\theta - \int_{C} \left|\frac{\partial u}{\partial r}\right|^2 r d\theta + \iint_{D_R} \frac{\partial u}{\partial r} \left[\frac{1}{r} \frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \bar{u}}{\partial \theta^2} + \bar{k}^2 \bar{u} \right] r dr d\theta$$

$$(3.14) \qquad + \iint_{D_R} \frac{\partial \bar{u}}{\partial r} \left[\frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + k^2 u \right] r dr d\theta = 0,$$

or

$$\int_{C_R} \left|\frac{\partial u}{\partial r}\right|^2 R\,d\theta - \int_{C} \left|\frac{\partial u}{\partial r}\right|^2 r\,d\theta + \iint_{D_R} \left|\frac{\partial u}{\partial r}\right|^2 dr\,d\theta + \iint_{D_R}\left(\frac{\partial u}{\partial r}\,\frac{\partial^2 \bar{u}}{\partial\theta^2} + \frac{\partial \bar{u}}{\partial r}\,\frac{\partial^2 u}{\partial\theta^2}\right)\frac{dr\,d\theta}{r}$$

$$+ \iint_{D_R}\left(k^2 u\,\frac{\partial \bar{u}}{\partial r} + \bar{k}^2\,\bar{u}\,\frac{\partial u}{\partial r}\right) r\,dr\,d\theta = 0. \tag{3.15}$$

Now integrating by parts with respect to θ, we have

$$\iint_{D_R}\left(\frac{\partial u}{\partial r}\,\frac{\partial^2 \bar{u}}{\partial\theta^2} + \frac{\partial \bar{u}}{\partial r}\,\frac{\partial^2 u}{\partial\theta^2}\right)\frac{dr\,d\theta}{r} = -\iint_{D_R}\left(\frac{\partial^2 u}{\partial r\partial\theta}\,\frac{\partial \bar{u}}{\partial\theta} + \frac{\partial^2 \bar{u}}{\partial r\partial\theta}\,\frac{\partial u}{\partial\theta}\right)\frac{dr\,d\theta}{r}$$

$$= -\iint_{D_R} \frac{\partial}{\partial r}\left|\frac{\partial u}{\partial\theta}\right|^2 \frac{dr\,d\theta}{r}$$

$$= -\iint_{C_R} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{d\theta}{R} + \int_{C} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{d\theta}{r}$$

$$- \iint_{D_R} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{dr\,d\theta}{r^2}\,. \tag{3.16}$$

Inserting this result into (15), we obtain

$$\int_{C} \left|\frac{\partial u}{\partial r}\right|^2 r\,d\theta = \int_{C_R} \left|\frac{\partial u}{\partial r}\right|^2 R\,d\theta - \int_{C_R} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{d\theta}{R} + \int_{C} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{d\theta}{r} + \iint_{D_R} \left|\frac{\partial u}{\partial r}\right|^2 dr\,d\theta$$

$$- \iint_{D_R} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{dr\,d\theta}{r^2} + \iint_{D_R}\left(k^2 u\,\frac{\partial \bar{u}}{\partial r} + \bar{k}^2\bar{u}\,\frac{\partial u}{\partial r}\right) r\,dr\,d\theta$$

$$\leq \int_{C_R} \left|\frac{\partial u}{\partial r}\right|^2 R\,d\theta + \int_{C} \left|\frac{\partial u}{\partial\theta}\right|^2 \frac{d\theta}{r} + \iint_{D_R} \left|\frac{\partial u}{\partial r}\right|^2 dr\,d\theta$$

$$+ \iint_{D_R}\left(k^2 u\,\frac{\partial \bar{u}}{\partial r} + \bar{k}^2\bar{u}\,\frac{\partial u}{\partial r}\right) r\,dr\,d\theta\,. \tag{3.17}$$

By making use of the radiation condition, C. Wilcox[21] has shown that

$$\lim_{R\to\infty}\Big[2\ \mathrm{Im}\ k\ |k|^2\iint_{D_R}|u|^2dS+2\ \mathrm{Im}\ k\iint_{D_R}|\nabla u|^2dS$$

$$+\int_{C_R}\left|\frac{\partial u}{\partial r}\right|^2ds+|k|^2\int_{C_R}|u|^2ds\Big]=2\ \mathrm{Im}k\int_C u\,\frac{\partial\bar{u}}{\partial r}\,ds. \tag{3.18}$$

Since all terms in the left member of (18) are positive, we obtain an inequality by dropping the last two terms of the left member. We then square this inequality. From the inequality which results, we drop the two squared terms of the left member and obtain

$$8|k|^2(\mathrm{Im}k)^2\iint_D|u|^2dS\iint_D|\nabla u|^2dS\le 4\Big[\mathrm{Im}\ k\int_C u\,\frac{\partial\bar{u}}{\partial r}\,ds\Big]^2. \tag{3.19}$$

Then

$$(2|k|^2)\Big(\iint_D|u|^2dS\iint_D|\nabla u|^2dS\Big)^{1/2}\le\frac{\sqrt{2}|k|}{\mathrm{Im}\ k}\left|\mathrm{Im}\ k\int_C u\,\frac{\partial\bar{u}}{\partial r}\,ds\right|$$

$$\le\frac{\sqrt{2}|k|^2}{\mathrm{Im}\ k}\Big(\int_C|u|^2ds\int_C\left|\frac{\partial u}{\partial r}\right|^2ds\Big)^{1/2} \tag{3.20}$$

$$\le\frac{|k|^4}{2(\mathrm{Im}k)^2\varepsilon^2}\int_C|u|^2ds+\varepsilon^2\int_C\left|\frac{\partial u}{\partial r}\right|^2ds,$$

where ε^2 is any positive number. (18) also implies

$$2\mathrm{Im}k\iint_D|\nabla u|^2dS+\lim_{R\to\infty}\int_{C_R}\left|\frac{\partial u}{\partial R}\right|^2ds\le\varepsilon^2\int_C\left|\frac{\partial u}{\partial r}\right|^2ds+\frac{(\mathrm{Im}k)^2}{\varepsilon^2}\int_C|u|^2ds. \tag{3.21}$$

If we now take the limit as R tends to infinity in (17), we may use the inequalities (20) and (21) in the resulting equation to obtain

$$\int_C\left|\frac{\partial u}{\partial r}\right|^2ds\le\int_C\frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2ds+\Big(\frac{|k|^4}{2(\mathrm{Im}k)^2}+(\mathrm{Im}k)^2\Big)\frac{1}{\varepsilon^2}\int_C|u|^2ds$$

$$+2\varepsilon^2\int_C\left|\frac{\partial u}{\partial r}\right|^2ds, \tag{3.22}$$

provided $2s\ \mathrm{Im}\ k\ge 1$. Then, provided $1-2\varepsilon^2>0$,

$$\int_C \left|\frac{\partial u}{\partial r}\right|^2 ds \le \frac{1}{1-2\varepsilon^2}\left[\int_C \frac{1}{r^2}\left|\frac{\partial u}{\partial\theta}\right|^2 ds + \left(\frac{|k|^4}{2(\mathrm{Im}k)^2} + (\mathrm{Im}k)^2\right)\frac{1}{\varepsilon^2}\int_C |u|^2 ds\right]. \tag{3.23}$$

This is the two-dimensional version of (5) and thus the Lemma is proved.

Proof of Lemma 3.2. The procedure for obtaining the estimate of Lemma 2 is to mimic the proof of Lemma 1 after having replaced the polar coordinates of Lemma 1 by the coordinate system referred to in the hypothesis of Lemma 2. In what follows, α and β as subscripts denote differentiation with respect to these variables.

In these new coordinates, the reduced wave equation is

$$FG\left[\left(\frac{F}{G}u_\alpha\right)_\alpha + \left(\frac{G}{F}u_\beta\right)_\beta\right] + k^2 u = 0. \tag{3.24}$$

In place of equation (17), we obtain

$$\begin{aligned}\int_B |u_\alpha|^2 \frac{F}{G}\,d\beta = \int_{B_R} |u_\alpha|^2 \frac{F}{G}\,d\beta - \int_{B_R} |u_\beta|^2 \frac{G}{F}\,d\beta + \int_B |u_\beta|^2 \frac{G}{F}\,d\beta \\ + \iint |u_\alpha|^2 \left(\frac{F}{G}\right)_\alpha d\alpha d\beta + \iint |u_\beta|^2 \left(\frac{G}{F}\right)_\beta d\alpha d\beta \\ + \iint (k^2\,\overline{u}_\alpha u + \overline{k}^2 u_\alpha \overline{u})\,\frac{d\alpha d\beta}{FG}\,.\end{aligned} \tag{3.25}$$

Here the domain of integration is the area between the curve B corresponding to $\alpha = 0$, say, and the curve B_R corresponding to $\alpha = R$. The hypothesis of orthogonality of the coordinate system implies that $\partial(x,y)/\partial(\alpha,\beta) = F^{-1}G^{-1}$. Then $d\alpha d\beta/FG = dS$ and $G^{-1}d\beta = ds$ along B. Since α is the length along the α-curves, $F \equiv 1$. Since B is a curve of constant α, $Fu_\alpha = \partial u/\partial n$ along B. Combining these facts and the remaining hypotheses, Wilcox' identity (18) enables us to obtain the estimate (4) of the lemma. The condition (3) of the hypothesis is necessary in order to apply Wilcox' identity.

When B is a convex curve, an appropriate orthogonal coordinate system is readily exhibited. Let B be given by

$$x = \xi(\beta) \qquad y = \eta(\beta), \tag{3.26}$$

where β is arclength along B, and $\xi(\beta)$ and $\eta(\beta)$ are twice differentiable. The curves β = constant are the normals to B and the curves α = constant are the parallels to B. Let $x = x(\alpha,\beta)$ and $y = y(\alpha,\beta)$ be the equation of the coordinate transformation, and let B correspond to $\alpha = 0$. Then

$$x = \alpha\eta_\beta + \xi \qquad y = -\alpha\xi_\beta + \eta. \tag{3.27}$$

Let $\mathcal{K} = \eta_{\beta\beta}\xi_{\beta} - \xi_{\beta\beta}\eta_{\beta} \geq 0$ be the curvature of B. Then

$$F^{-1} = 1 \qquad \text{and} \qquad G^{-1} = 1 + a\mathcal{K} . \tag{3.28}$$

From (28) we see that the condition (3) becomes

$$\operatorname{Im} k \geq \left| \frac{\mathcal{K}}{1+a\mathcal{K}} \right| \geq \mathcal{K} . \tag{3.29}$$

Since 1/a is the curvature of a circle of radius a, (29) is seen to be analogous to the corresponding condition of Lemma 1.

Proof of Lemma 3.3. For k real we have

$$\begin{aligned} \iint_{D_R} \left(k^2 u \frac{\partial \bar{u}}{\partial r} + \bar{k}^2 \bar{u} \frac{\partial u}{\partial r}\right) r\,dr\,d\theta &= k^2 \iint_{D_R} \frac{\partial |u|^2}{\partial r}\, r\,dr\,d\theta \\ &= k^2 \int_{C_R} |u|^2 R\,d\theta - k^2 \int_C |u|^2 r\,d\theta - k^2 \iint_{D_R} |u|^2 dr\,d\theta . \end{aligned} \tag{3.30}$$

Inserting this into (17), we obtain

$$\begin{aligned} \int_C \left|\frac{\partial u}{\partial r}\right|^2 r\,d\theta \leq \int_{C_R} \left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2 |u|^2 \right) R\,d\theta + \int_C \frac{1}{r^2} \left|\frac{\partial u}{\partial \theta}\right|^2 r\,d\theta \\ + \iint_{D_R} \left(\left|\frac{\partial u}{\partial r}\right|^2 - k^2 |u|^2 \right) dr\,d\theta - k^2 \int_C |u|^2 r\,d\theta . \end{aligned} \tag{3.31}$$

From the radiation condition (1), we see that the first term in the right member of (31) is equal to

$$ik \int_{C_R} \left(u \frac{\partial \bar{u}}{\partial R} - \bar{u} \frac{\partial u}{\partial R}\right) R\,d\theta$$

when R tends to infinity. From Green's theorem, we see that this latter quantity is equal to

$$ik \int_C \left(u \frac{\partial \bar{u}}{\partial n} - \bar{u} \frac{\partial u}{\partial n}\right) r\,d\theta .$$

Thus

$$\left| \lim_{R \to \infty} \int_{C_R} \left(\left|\frac{\partial u}{\partial r}\right|^2 + k^2 |u|^2 \right) R\,d\theta \right| \leq \varepsilon^2 \int_C \left|\frac{\partial u}{\partial r}\right|^2 ds + \frac{|k|^2}{\varepsilon^2} \int_C |u|^2 ds . \tag{3.32}$$

This provides us with an estimate for the first term of the right member of (31).

We will now show that for ka sufficiently large

$$(3.33)\qquad \iint_D \left(\left|\frac{\partial u}{\partial r}\right|^2 - k^2|u|^2 \right) dr d\theta - k^2 \int_C |u|^2 r d\theta < 0.$$

This will permit us to neglect the last two terms of the right member of (31). Thus (32) and (33) will provide us with our desired estimate.

Let u = f(θ) on C. Then

$$(3.34)\qquad u = \sum_{-\infty}^{\infty} a_n H_n^{(1)}(kr)e^{in\theta},$$

where

$$(3.35)\qquad a_n H_n^{(1)}(ka) = \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(\theta) d\theta.$$

We have from Parseval's equality

$$(3.36)\qquad \int_0^{2\pi} |u|^2 d\theta = 2\pi \sum_{-\infty}^{\infty} |a_n|^2 \, |H_n^{(1)}(kr)|^2$$

and

$$\int_0^{2\pi} |u_r|^2 d\theta = 2\pi k^2 \sum_{-\infty}^{\infty} |a_n|^2 \, |H_n^{(1)\prime}(kr)|^2 .$$

Here the prime denotes differentiation. Now

$$\iint_D \left(\left|\frac{\partial u}{\partial r}\right|^2 - k^2|u|^2 \right) dr d\theta - k^2 \int_C |u|^2 r d\theta$$

$$(3.37)\qquad = 2\pi k^2 \sum_{-\infty}^{\infty} |a_n|^2 \left[\int_a^{\infty} \left(|H_n^{(1)\prime}(kr)|^2 - |H_n^{(1)}(kr)|^2 \right) dr - a|H_n^{(1)}(ka)|^2 \right]$$

$$= 2\pi k \sum_{-\infty}^{\infty} |a_n|^2 \left[\int_{ka}^{\infty} \left(|H_n^{(1)\prime}(\tau)|^2 - |H_n^{(1)}(\tau)|^2 \right) d\tau - ka|H_n^{(1)}(ka)|^2 \right] .$$

We will show in Lemma 4 below that the quantity

$$(3.38)\qquad F(k,n) = \int_k^{\infty} \left(|H_n^{(1)\prime}(\tau)|^2 - |H_n^{(1)}(\tau)|^2 \right) d\tau - k|H_n^{(1)}(k)|^2$$

is negative for k larger than some fixed constant uniformly for all n. Thus Lemma 4 and (37) will imply the assertion of (33). Then inserting (32) into (31), we obtain

$$(3.39) \qquad \int_C \left|\frac{\partial u}{\partial r}\right|^2 a d\theta \leq \frac{1}{1-\varepsilon^2}\left[\frac{|k|^2}{\varepsilon^2}\int_C |u|^2 a d\theta + \int_C \frac{1}{a^2}\left|\frac{\partial u}{\partial \theta}\right|^2 a d\theta\right]$$

which proves Lemma 3.

We now turn our attention to

Lemma 3.4: There exists a constant K such that if $k > K$ then $F(k,n) < 0$ for all n.

Proof of Lemma 3.4. The method of proof is straightforward but the details are tedious. We will therefore sketch the ideas of the proof.

By integrating the term

$$\int_k^\infty H_n^{(1)\prime}(\tau)\, \overline{H_n^{(1)\prime}(\tau)}\, d\tau$$

by parts and applying Bessel's equation, we find that

$$(3.40) \qquad \begin{aligned} F(k,n) = &-\left(\frac{1}{2k} + k\right)\left|H_n^{(1)}(k)\right|^2 - \frac{1}{2}\frac{d}{dk}\left|H_n^{(1)}(k)\right|^2 \\ &+ \int_k^\infty \frac{\frac{1}{2} - n^2}{\tau^2}\left|H_n^{(1)}(\tau)\right|^2 d\tau . \end{aligned}$$

Now (cf. [2], p. 54)

$$(3.41) \qquad \left|H_n^{(1)}(Z)\right|^2 = \frac{8}{\pi^2}\int_0^\infty K_0(2Z \sinh t)\cosh 2nt\, dt, \qquad \text{Re } Z > 0.$$

(41) and (40) show that $F(k,n)$ is an even function of n. We may therefore confine our attention to non-negative values of n. Now $H_n^{(1)}(k) = O(1/\sqrt{k})$. Then $k|H_n^{(1)}(k)|^2 = O(1)$, and indeed $\lim_{k \to \infty} k|H_n^{(1)}(k)|^2$ is not zero (cf. [2], p. 85). The integral in (38) tends to zero as ka tends to infinity, while $ka|H_n^{(1)}(ka)|^2$ tends to a positive constant as we have just remarked. Thus $F(k,n)$ becomes negative for $k > K(n)$. We must now show that the $\max_{n \geq 0} K(n)$ is finite.

From (33) we compute

$$(3.42) \qquad \frac{dF(k,n)}{dk} = -|H_n'(k)|^2 - k\frac{d}{dk}|H_n(k)|^2 .$$

Now we use the asymptotic expansion (cf. [2], p. 87)

$$(3.43)\qquad H_n^{(1)}(k) = -i\,\frac{2^{1/2}}{\pi}\,(k^2-n^2)^{-1/4}\exp\left[-(n^2-k^2)^{1/2}+n\cosh^{-1}\frac{n}{k}\right]\left(\sqrt{\pi}+O\left(\frac{1}{k}\right)\right),$$

valid for k large and $n > k$. Inserting (43) into (42) we find that $\frac{dF(k,n)}{dk} < 0$ for $n-1 > k$.

Now $|H_n^{(1)}(\tau)|^2$ is a monotone decreasing function of τ for $\tau < 0$ and $n > \frac{1}{2}$ (cf. [12], p.35). Then for $A > k$

$$(3.44)\qquad \int_k^{\infty}\frac{\frac{1}{2}-n^2}{\tau^2}\,|H_n^{(1)}(\tau)|^2 d\tau \leq \left(\frac{1}{2}-n^2\right)|H_n^{(1)}(A)|^2\int_k^{A}\frac{d\tau}{\tau^2}\,.$$

We insert this estimate into the expression (40) for F(k,n), and show, using (43), that $F(k,n) < 0$ for $n > 2k$. We repeat the procedure with the asymptotic expansion ([2], p. 86)

$$(3.45)\qquad H_n^{(1)}(k) = \frac{2^{1/2}}{\pi}\,(k^2-n^2)^{-1/4}\exp\left[i(k^2-n^2)^{1/2}+in\sin^{-1}\frac{n}{k}+i\,\frac{\pi}{2}\left(n+\frac{1}{2}\right)\right]\cdot\left(\sqrt{\pi}+O\left(\frac{1}{k}\right)\right)$$

valid for k large and $n < k$. In this manner we find that $F(k,n) < 0$ for $n < k$ and n sufficiently large. The least value of n for which this is true does not increase with k since the formula (45) becomes a better approximation, the larger k is. Since $F(k,n) < 0$ for $k > K(n)$, the last remarks show that $F(k,n) < 0$ for all $n < k$, for k sufficiently large. Since $\frac{dF(k,n)}{dk} < 0$ for $n > k+1$ and $F(k,n) < 0$ for $n > 2k$ for some sufficiently large k, we may, by increasing k, obtain $F(k,n) < 0$ for $n > k+1$. Thus for a fixed large k, $F(k,n) < 0$ for all $n < k$ and all $n > k+1$. Now (cf. [21], p. 231)

$$(3.46)\qquad H_n^{(1)}(n) \sim \frac{\gamma}{n^{1/3}}$$

where γ is a numerical constant. Using this formula in (40) we find that $F(n,n) < 0$ for n sufficiently large. By studying the derivation of (46), we find that the same formula is valid for $H_{n+1}^{(1)}(n)$. Thus $F(n+1,n) < 0$ for n sufficiently large. Thus $F(k,n) < 0$ for all n for all k larger than a large constant K, and Lemma 4 is proved.

4. Parametric theory of solutions of the reduced wave equation $\Delta u + k^2 u = 0$

The parametric theory of ordinary differential equations has been developed to a high level. In particular the asymptotic theory has been studied extensively by many authors including G. D. Birkhoff, Langer and others. On the other hand the parametric theory of partial differential equations has barely begun to be investigated.

The existing literature on this subject falls into several categories. In one of these categories an appropriate limit partial differential equation is associated with an original partial differential equation is associated with an original partial differential equation. The original equation contains a parameter which is considered large. The limit equation is usually, but not necessarily, the formal limit of the original equation as the parameter tends to infinity. A statement is then made about the difference between the solution of the original equation and an appropriate solution of the limit equation as the parameter tends to infinity. In 1922 W. Sternberg[17] considered the equations

$$\frac{\partial^2 u}{\partial x \partial y} + \left(C(x,y) + k^2\right) u = 0 \tag{4.1}$$

and

$$\frac{\partial^2 w}{\partial x \partial y} + k^2 w = 0. \tag{4.2}$$

Sternberg shows that

$$u - w = 0\left(\frac{1}{k^{1/3}}\right),$$

where u is a solution of (1) and w is an appropriate solution of (2). In a second paper Sternberg[17] shows that $u-w = 0\left(\frac{1}{k^{2/5}}\right)$ for the pair of equations

$$\Delta u + \left(A(x,y;k) + k^2\right) u = 0$$

$$\Delta w + k^2 w = 0.$$

Here A is bounded uniformly with respect to k. In 1944 W. Wasow[20] considered the pair of equations

$$\Delta u + k \frac{\partial u}{\partial x} = k\, f(x,y)$$

$$\frac{\partial w}{\partial x} = f(x,y).$$

In 1950 N. Levinson[9] considered the pair of equations

$$\frac{1}{k} \Delta u + A(x,y) \frac{\partial u}{\partial x} + B(x,y) \frac{\partial u}{\partial y} + C(x,y)u = D(x,y)$$

$$A(x,y) \frac{\partial w}{\partial x} + B(x,y) \frac{\partial w}{\partial y} + C(x,y)w = D(x,y).$$

In all of these results the domain of definition D of the solutions is bounded. In Sternberg's results there is no discussion of boundary values of the solutions. In the results of Wasow and Levinson it is shown that the solution w of the limit equation takes on the values of the solution u of the original equation on appropriate segments of the boundary B of D. This is as much as can be expected since the original and limit equations are of different orders.

A second category of the literature deals with formal series expansions of solutions. In 1933 G. D. Birkhoff[3] suggested a procedure for obtaining formal series expansions of solutions of the reduced wave equation

$$(4.3) \qquad \Delta u + k^2 u = 0.$$

The procedure is to insert the series $e^{ik\Psi} \sum_{0}^{\infty} \frac{\sigma_n}{(ik)^n}$ into (3), and equate coefficients of the various powers of k to zero. This gives rise to a recursive system of first-order partial differential equations for the determination of the coefficients Ψ and σ_n, of the terms of the series. Birkhoff conjectured that there exists an appropriate solution of (3) to which the formal series is asymptotic.

In 1944 W. Trjitzinsky[18] generalized Birkhoff's procedure by considering a more complicated formal series and a linear partial differential equation of the m-th order. The domain D of definition of the solution of the equation is bounded. Trjitzinsky shows that under appropriate conditions, the formal series exists in a subdomain D_1 of D. In the case when the equation is of elliptic or hyperbolic type, he also shows that there exists an appropriate solution of the differential equation to which the formal series is asymptotic in a subdomain D_2 of D_1.

In 1952 M. Kline[8] considered the hyperbolic wave equation $\Delta u = \frac{\partial^2 u}{\partial t^2}$ from which $\Delta u + k^2 u = 0$ is reduced. He showed that if a solution of $\Delta u = \frac{\partial^2 u}{\partial t^2}$ has certain jump discontinuities, then the corresponding solution of $\Delta u + k^2 u = 0$ possesses an asymptotic expansion of the type conjectured by Birkhoff.

In 1955 J. B. Keller, R. M. Lewis and B. D. Seckler[7] applied the procedure of Birkhoff to a large number of specific diffraction problems. They compare their series with those obtained from the exact solution of these problems. In every such comparison there is complete agreement between the known correct result and the formal series.

We will investigate further the range of the validity of the method suggested by Birkhoff. We consider a solution u of the reduced wave equation defined in the exterior D of a bounded or unbounded domain. The values of u on the boundary B of D are prescribed. u satisfies the radiation condition.

We first illustrate the method of Birkhoff and show how to obtain a formal series expansion for u, following closely some of the material in [7]. Then we define the notion of regular domain and prove two lemmas upon which the proofs of the theorems given later will depend. Next, we discuss the asymptotic properties of the formal expansion. We show that the expansion is asymptotic to a solution of the reduced wave equation. Some theorems are given which exhibit the relationship between this solution and the exact solution of the boundary-value problem. In the case where the domain is regular and the complement of a certain class of bounded domains, we use the estimates of Section 3 to show that the formal expansion is asymptotic to the exact solution of the boundary-value problem.

We proceed now to review Birkhoff's method by applying it to a two-dimensional boundary-value problem. This problem may be stated as follows. Let u(x,y) be a solution of the reduced wave equation

$$\Delta u + k^2 u = 0 \tag{4.3}$$

in the exterior D of a bounded or unbounded domain. Let B, the boundary of D, be given parametrically by $x = x_o(t)$, $y = y_o(t)$. t is arclength along B. As t increases B is described in the usual positive sense. On B let u be prescribed by[*]

$$u_o = e^{ik\phi(x,y)} \sum_{n=o}^{M} \frac{a_n(x,y)}{(ik)^n} . \tag{4.4}$$

Finally let u satisfy the radiation condition

$$\lim_{R \to \infty} \int_{C_R} \left| \frac{\partial u}{\partial r} - iku \right|^2 ds = 0. \tag{4.5}$$

C_R is a circle of radius R with some fixed point in space as center. ds is a line element.

To find the asymptotic expansion of u, the procedure of Birkhoff is as follows: Assume that u has an asymptotic expansion u^{∞} of the form

---

[*] When it is necessary to stress the fact that a function is evaluated on the boundary B of D, we will use the subscript o.

$$u^{\infty} = e^{ik\Psi(x,y)} \sum_{n=o}^{\infty} \frac{\sigma_n(x,y)}{(ik)^n} . \tag{4.6}$$

We insert u^{∞} into (3), and equate the coefficients of various powers of k to zero. This yields the following system of equations for Ψ and σ_n, $n = 0,1,2,\ldots$:

$$(\nabla\Psi)^2 = 1 \tag{4.7}$$

$$2\nabla\sigma_n \cdot \nabla\Psi + \sigma_n\nabla^2\Psi = -\nabla^2\sigma_{n-1}, \qquad \sigma_{-1} = 0. \tag{4.8}$$

Thus Ψ and σ_n are solutions of first-order partial differential equations. Their boundary values are obtained by equating the value of the formal expansion on B to the boundary values (4), of u. We obtain

$$\Psi_o = \phi \tag{4.9}$$

$$\begin{aligned} \sigma_{n_o} &= a_n & \qquad n &= 0, 1, \ldots, M \\ \sigma_{n_o} &= 0 & \qquad n &= M+1, M+2, \ldots . \end{aligned} \tag{4.10}$$

Thus the formal expansion u^{∞} is equal to u on B.

To solve the eiconal equation (7), we let $\Psi_x = p$, $\Psi_y = q$ and use the method of characteristics. We denote the arclength along the characteristics by s. This method yields the following system of ordinary differential equations:

$$\begin{aligned} &a)\ \frac{dx}{ds} = p &\qquad &d)\ \frac{dp}{ds} = 0 \\ &b)\ \frac{dy}{ds} = q &\qquad &e)\ \frac{dq}{ds} = 0 \\ &c)\ \frac{d\Psi}{ds} = 1 &\qquad &f)\ p_o^2 + q_o^2 = 1 \end{aligned} \tag{4.11}$$

$$g)\ \frac{d\phi}{dt} = p_o \frac{dx_o}{dt} + q_o \frac{dy_o}{dt} .$$

From (11f) and (11g) we find

$$\begin{aligned} &a)\ p_o = \frac{d\phi}{dt} \frac{dx_o}{dt} \mp \frac{dy_o}{dt} \sqrt{1 - (\frac{d\phi}{dt})^2} , \\ &b)\ q_o = \frac{d\phi}{dt} \frac{dy_o}{dt} \pm \frac{dz_o}{dt} \sqrt{1 - (\frac{d\phi}{dt})^2} . \end{aligned} \tag{4.12}$$

From (11a)-(11e) we find

$$\begin{aligned} &a)\ x = p_o s + x_o \\ &b)\ y = q_o s + y_o \end{aligned} \tag{4.13}$$

for the characteristics or rays, and

$$(4.14) \qquad \Psi = s + \phi.$$

The recursion equations (8) may be solved as follows. From (7) we see that the vector $\nabla\Psi$ normal to the surfaces Ψ = constant is a unit vector. From (11a) and (11b) we see that the rays are also normal to this surface. Thus $\nabla\sigma_n \cdot \nabla\Psi$ is just the ordinary derivative $\frac{d\sigma_n}{ds}$ of σ_n with respect to the length of a ray along each ray. Thus (8) is just an ordinary differential equation along each ray. The solution of (8) along a ray is

$$(4.15) \qquad \sigma_n(s) = \sigma_n(0)\exp\left[-\frac{1}{2}\int_0^s \nabla^2\Psi dt\right] - \frac{1}{2}\int_0^s \exp\left[-\frac{1}{2}\int_t^s \nabla^2\Psi ds'\right] \nabla^2\sigma_{n-1}(t)dt$$

or

$$(4.16) \qquad \sigma_n(s) = \sigma_n(0)\left[\frac{G(s)}{G(0)}\right]^{1/2} - \frac{G^{1/2}(s)}{2}\int_0^s G^{-1/2}(t)\,\nabla^2\sigma_{n-1}(t)dt.$$

$G(s)$ is the curvature of the surface Ψ = constant at the point s along the ray emanating from $\left(x_0(t), y_0(t)\right)$, and $\sigma_n(0) = a_n\left(x_0(t), y_0(t)\right)$.

We will introduce new coordinates (s', β) in place of (s, t). s' is the distance along a ray measured from the envelope or caustic of the rays. β is the angle which each ray makes with the x-axis. These coordinates are called caustic coordinates. Let $s_0(t)$ be the distance along a ray from the caustic to B. $s_0(t)$ has the same sense as s. Then (s', β) are related to (s, t) by

$$(4.17) \qquad s' = s + s_0(t), \qquad \beta = \tan^{-1}\frac{q_0(t)}{p_0(t)}.$$

The regularity of the caustic coordinates depends on B and ϕ. We will now drop the prime. In the new coordinates (16) and (14) become respectively

$$(4.18) \qquad \sigma_n(s,\beta) = \sigma_n(s_0,\beta)\left(\frac{s_0}{s}\right)^{1/2} - \frac{1}{2s^{1/2}}\int_{s_0}^s t^{1/2}\,\nabla\,\sigma_{n-1}(t,\beta)dt,$$

$$(4.19) \qquad \Psi = s - s_0 - \phi.$$

Let the caustic be given by

$$(4.20) \qquad x = \xi(\beta), \qquad y = \eta(\beta).$$

Let a dot denote differentiation with respect to β. Let

$$\begin{aligned} a &= \dot{\xi}^2 + \dot{\eta}^2 \\ b &= \dot{\xi}\cos\beta + \dot{\eta}\sin\beta \qquad (4.21) \\ c &= \ddot{\xi}\cos\beta + \ddot{\eta}\sin\beta . \end{aligned}$$

With these abbreviations, we may express the σ_n in caustic coordinates as follows*

$$(4.22) \qquad \sigma_n(s,\beta) = \sum_{j=o}^{3n} f_{jn}(\beta) s^{-1/2-j}$$

$$(4.23) \qquad \begin{aligned} f_{jn} &= \frac{1}{2j}\Big\{(j-\tfrac{1}{2})f_{j-1,n-1} + \ddot{f}_{j-1,n-1} + 2b(j-1)\dot{f}_{j-2,n-1} \\ &\qquad + c(j-\tfrac{3}{2})f_{j-2,n-1} + a(j-\tfrac{5}{2})(j-\tfrac{1}{2})f_{j-3,n-1}\Big\}, \quad j \neq 0 \\ f_{on} &= \sigma_n(s_o,\beta)s_o^{1/2} - \sum_{j=1}^{3n} f_{jn}s_o^{-j}, \qquad n \neq 0 \\ f_{oo} &= \sigma_o(s_o,\beta)s_o^{1/2} . \end{aligned}$$

We now turn to a discussion of the formal details concerned with the existence of the formal series and the regularity of the caustic coordinate system. We make the following definitions and assumptions.

p_o and q_o will be real for all values of t provided the radicals in (12) are real. This requires $(\frac{d\phi}{dt})^2 \leq 1$. If the boundary values of u arise from an incident field, this condition is automatically satisfied. For in this case, the phase of the incident field satisfies the eiconal equation (7). From (13) and the ambiguity of sign in (12), we see that each value of t gives rise to two rays. Thus as s increases from s_o, two rays will emanate from B and enter D or its complement. In principle we could use either sign and the corresponding ray to solve for the coefficients Ψ and σ_n, n = 0, 1, 2, ..., of the formal series. Indeed we could use one sign for certain values of t, and the other for the remaining values.

Now the characteristic differential equations and the choice of sign in (12) assign a positive sense to the arclength s on each ray. If as s increases from s_o, a ray enters D, then along this ray the formal expansion behaves like $e^{ik|s|}$. Since a solution of $\triangle u + k^2 u = 0$ which satisfies the radiation condition behaves like e^{ikr}, the formal expansion could not be valid along the second type of ray. If for each value of t one of the rays emanating from B were of the former type while the second were of the latter type, this observation would serve to determine a single ray for each t along which to compute Ψ and the σ_n. The following lemma and remark assert that this is indeed the case.

---

* For details concerning the derivation of these expressions, see [7].

Lemma. If $\frac{\partial(x,y)}{\partial(s,t)} > 0$ for a value of t, then the corresponding ray enters D as s increases from s_o, and conversely.

Remark: $\frac{\partial(x,y)}{\partial(s,t)} = \mp\sqrt{1 - (\frac{\partial\phi}{\partial t})^2}$, the upper and lower signs corresponding to the upper and lower signs in (12).

The proofs of these facts follow from simple computations using (11), (12) and (13), and will therefore not be given.

If $\frac{\partial(x,y)}{\partial(s,t)} = 0$ for $s = s_o$ and certain values of t, the preceding remark asserts that the radicals in (12) vanish. In this case the ambiguity in sign disappears.

We introduce the notion of a regular domain in the following

Definition. A domain D with boundary B is regular relative to ϕ, if the portions of all rays for $s > s_o$ form a field in D, if $s_o \geq \varepsilon > 0$, and if

$$\max_{\beta}\left\{ |\xi(\beta)|\,|\eta(\beta)|\,|\dot{\xi}(\beta)|\,|\dot{\eta}(\beta)|\,|\ddot{\xi}(\beta)|\,|\ddot{\eta}(\beta)| \right\} < A.$$

ε and A are positive numbers.

Hereafter we will consider only the portions of the rays which correspond to positive values of s.

Examples of Regular Domains

i) The domain to the left of the parabola $x = y^2$ is regular relative to the phase $\phi = x$ (plane wave). The caustic in this case degenerates to the focus of the parabola.

ii) The domain exterior to a bounded smooth convex curve is regular relative to ϕ = constant.

iii) No domain which is the exterior of a bounded domain is regular relative to $\phi = x$. If the boundary is smooth, then by deleting appropriate portions of D a regular subdomain of D is obtained.

From (22) and (23) we see that $\sigma_n(s,\beta)$ is a function of

$$a_o, \frac{\partial a_o}{\partial\beta}, \ldots, \frac{\partial^{2n} a_o}{\partial\beta^{2n}}, a_1, \frac{\partial a_1}{\partial\beta}, \ldots, \frac{\partial^{2n-2} a_1}{\partial\beta^{2n-2}}, \ldots, a_n . \qquad (4.24)$$

Thus the amplitudes a_n, $n = 0, 1, \ldots, N$ of the boundary values of u must be sufficiently often differentiable for the formal expansion to exist to N terms.

Hereafter we will assume that these quantities are as differentiable as is necessary to justify the arguments which follow.

The relationship between caustic and cartesian coordinates is

$$(4.25) \qquad x = \xi(\beta) + s\cos\beta, \qquad y = \eta(\beta) + s\sin\beta.$$

The Jacobian of this transformation is

$$(4.26) \qquad J(s,\beta) = \dot{\xi}\sin\beta - \dot{\eta}\cos\beta - s.$$

The Laplacian in caustic coordinates is

$$(4.27) \qquad \nabla^2 = \left(1+\frac{a}{s^2}\right)\frac{\partial^2}{\partial s^2} + \left(\frac{1}{s} - \frac{c}{s^2} - \frac{a}{s^3}\right)\frac{\partial}{\partial s} - \frac{2b}{s^2}\frac{\partial^2}{\partial s\partial\beta} + \frac{1}{s^2}\frac{\partial^2}{\partial\beta^2} + \frac{b}{s^3}\frac{\partial}{\partial\beta}.$$

a, b and c are defined in (21). We will now state and prove

Lemma 4.1. Let D be regular relative to $\emptyset$. Let the quantities occurring in (24) and the first and second derivatives of these quantities be bounded. Then

$$(4.28) \qquad |\Delta\sigma_n| \quad \text{and} \quad \iint_D |\Delta\sigma_n|\,dS$$

are bounded.

Proof. Applying (27) to (22), we obtain

$$(4.29) \qquad \nabla^2\sigma_n = \frac{1}{s^{5/2}}\sum_{j=0}^{3n}\frac{1}{s^j}\Bigg[(\tfrac{1}{2}+j)^2 f_{jn} + \ddot{f}_{jn} + c(\tfrac{1}{2}+j)f_{jn} + \frac{c(\frac{1}{2}+j)f_{jn} + 2(1+j)b\dot{f}_{jn}}{s} + \frac{(\frac{1}{2}+j)(\frac{5}{2}+j)af_{jn}}{s^2}\Bigg].$$

Since D is regular relative to $\emptyset$, $s \geq \varepsilon > C$ at every point in D. From the recurrence formulas (23) and the hypotheses we see that all quantities in the right member of (29) are bounded. Thus $|\nabla^2\sigma_n|$ is bounded in D. From (29) we have a constant $C_1(n)$ such that

$$(4.30) \qquad |\nabla^2\sigma_n| \leq \frac{C_1(n)}{s^{5/2}}.$$

Then

$$(4.31) \qquad \iint_D |\nabla\sigma_n|\,dxdy \leq \int_0^{2\pi}\int_{s_0}^{\infty}\frac{C_1(n)}{s^{5/2}}\,|J(s,\beta)|\,dsd\beta \leq C(n)\int_0^{2\pi}\int_{s_0}^{\infty}\frac{d\beta ds}{s^{3/2}},$$

from (26). Here C(n) is some constant. This proves the lemma.

We will now state and prove

Lemma 4.2. Under the hypothesis of Lemma 4.1, u^N satisfies the radiation condition.

Proof. We have

$$u^N = e^{ik(\phi - s_o)} \frac{e^{iks}}{s^{1/2}} \sum_{n=o}^{N} \frac{\sum_{j=o}^{3n} f_{jn} s^{-j}}{(ik)^n} . \tag{4.32}$$

Now, denoting differentiation with respect to r, β, and s by subscripts, we have from (32)

$$\begin{aligned} u^N_r - iku^N &= u^N_\beta \beta_r + u^N_s s_r - iku^N \\ &= \frac{e^{iks}}{s^{1/2}} \sum_{n=o}^{N} \frac{\sum_{j=o}^{3n} \left(e^{ik(\phi - s_o)} f_{jn}\right)_\beta s^{-j}}{(ik)^n} \beta_r \\ &+ \frac{ike^{iks}}{s^{1/2}} \sum_{n=o}^{N} \frac{\sum_{j=o}^{3n} \left(e^{ik(\phi - s_o)} f_{jn}\right) s^{-j}}{(ik)^n} (s_r - 1) \\ &+ \frac{e^{iks}}{s^{3/2}} \sum_{n=o}^{N} \frac{\sum_{j=o}^{3n} \left(e^{ik(\phi - s_o)} f_{jn}\right) s^{-j}}{(ik)^n} s_r . \end{aligned} \tag{4.33}$$

From (25) we have

$$r^2 = \xi^2 + \eta^2 + s^2 + 2s(\eta \sin \beta + \xi \cos \beta) \tag{4.34}$$

and

$$\tan \theta = \frac{\eta + s \sin \beta}{\xi + s \cos \beta} . \tag{4.35}$$

From (34) we have

$$\lim_{s \to \infty} \frac{r}{s} = 1. \tag{4.36}$$

Differentiating (34) and (35) with respect to r, we obtain

$$\begin{aligned} r &= \beta_r\left(\xi\dot{\xi} + \eta\dot{\eta} + s(\dot{\eta} \sin \beta + \eta \cos \beta + \dot{\xi} \cos \beta - \xi \sin \beta)\right) \\ &\qquad + s_r(s + \eta \sin \beta + \xi \cos \beta), \\ 0 &= \beta_r \left((\dot{\eta} + s \cos \beta)(\xi + s \cos \beta) - (\dot{\xi} - s \sin \beta)(\eta + s \sin \beta)\right) \\ &\qquad + s_r \left(\sin \beta(\xi + s \cos \beta) - \cos \beta(\eta + s \sin \beta)\right) . \end{aligned} \tag{4.37}$$

Upon solving (37) for β_r and s_r, we may deduce, using (36), that

$$(4.38) \qquad \beta_r = O(\tfrac{1}{s}) \qquad \text{and} \qquad s_r \to 1 \quad \text{as} \quad s \to \infty.$$

Then from (33) we see that

$$(4.39) \qquad |u_r^N - iku^N| \leq o(1)\, O\left(\frac{1}{s^{1/2}}\right) = o(1)\, O\left(\frac{1}{r^{1/2}}\right)$$

which proves the lemma.

We are now in a position to discuss the asymptotic properties of the formal series. First we will consider domains D which are regular relative to $\emptyset$ and whose boundaries B are bounded. These conditions will not be mentioned specifically in the hypotheses of the theorems which are stated. Later we will indicate the extent to which these results are valid when B is unbounded but D is still regular. Finally we consider the case where D is not relative to $\emptyset$.

We will prove the following theorems.

Theorem 4.1. u^N, the formal series to N terms, approximates a solution of the reduced wave equation to the order $O(k^{-N-2/5})$. This solution is the exact solution of the boundary-value problem plus a remainder. The boundary values of this remainder are at least as small as the above degree of approximation.

Theorem 4.2. u^N approximates to the order $O(k^{-N-2/5})$ a solution of the reduced wave equation composed of a single- and double-layer distribution. The densities of these distributions are, respectively, the values of the normal derivative of u^N on B and the value of u^N on B.

We know that the boundary values of u^N are the same as the boundary values of u for $N \geq M$. For $N < M$ we know from (10) that $|u_0^N - u_0| \leq O(k^{-N-1})$. The following relationship persists between the values of the normal derivative of u^N on B, $\partial u_0^N/\partial n$, and the values of the normal derivative of u on B, $\partial u_0/\partial n$.

Theorem 4.3. $\partial u_0^N/\partial n$ satifies the integral equation of the first kind for $\partial u_0/\partial n$ to the order $O(k^{-N-2/5})$ for $N \geq M$ or to the order $O(k^{-N-1/3})$ for $N < M$.

Further we shall prove

Theorem 4.4. Let B be a smooth simply closed curve, and let Im k > 0. If 2 Im k is larger than some constant C(B), then

$$u^N = u + \frac{O(|k|^{-N+19/15})}{(\operatorname{Im} k)^2}.$$

Theorem 4.5. Let B be a circle of radius a, let k be real, and let $k > \frac{1}{a}$.

Then

$$u^N = u + O(k^{-N + 3/5}).$$

For the proofs of all these theorems we will need the following

Lemma 4.3. Let w satisfy the radiation condition and be a solution of $\triangle u + k^2 u = f$. Let $|f|$ and $\iint_D |f| dS$ be bounded uniformly with respect to k. Then for P in the closure of D, we have

$$(4.40) \qquad \left| w(P) + \frac{i}{4} \int_B \left[\frac{\partial w}{\partial n} H_o^{(1)}(k|P-Q|) - w \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|) \right] ds \right| \leq O(k^{-2/5}).$$

Proof of Lemma 4.3. Since w satisfies the radiation condition and $\iint_D |f| dS$ exists, we have from the corollary of Theorem 1.3

$$(4.41) \qquad \begin{aligned} 4iw(P) = & \int_B \left[\frac{\partial w}{\partial n} H_o^{(1)}(k|P-Q|) - w \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|) \right] ds \\ & - \iint_D f H_o^{(1)}(k|P-Q|) dS. \end{aligned}$$

We will now show that the domain integral is bounded by $O(k^{-2/5})$. The Hankel function decays like $(k|P-Q|)^{-1/2}$ as $k|P-Q| \to \infty$, and has an integrable logarithmic singularity at P = Q. These facts together with the hypotheses show that the domain integral is absolutely integrable. Now break up D into a bounded part containing P and its complement. In the unbounded part we have $|P-Q| >$ constant. Thus the integral over this part is bounded by $O(k^{-1/2})$. Sternberg[17] has shown that

$$(4.42) \qquad \iint_A |H_o^{(1)}(k|P-Q|)| dS \leq O(k^{-2/5}),$$

where A is a bounded domain and P is any point in the closure of A. The constant occurring in the order relation (42) depends on the area of A. Since $|f|$ is bounded uniformly with respect to k, (42) completes the proof of the lemma.

Now we proceed to prove Theorems 1-5.

Proof of Theorem 4.1. Let

$$(4.43) \qquad k^{-N} w^N = u - u^N.$$

From the recurrence equations (7) and (8), we obtain

$$(4.44) \qquad \triangle w^N + k^2 w^N = (-i)^N e^{ik\psi} \triangle \sigma_N .$$

Lemma 1 implies that $f = (-i)^N e^{ik\psi} \Delta \sigma_N$ satisfies the hypothesis of Lemma 3. Lemma 2 implies that w^N satisfies the radiation condition. Thus if

$$(4.45) \qquad 4ik^N R^N = \int_B \left[\frac{\partial w^N}{\partial n} H_o^{(1)}(k|P-Q|) - w^N \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|) \right] ds,$$

Lemma 3 applied to w^N implies

$$(4.46) \qquad |w^N - k^N R^N| \leq O(k^{-2/5}).$$

By (43) this is

$$(4.47) \qquad |u - u^N - R^N| \leq O(k^{-N-2/5}).$$

Since R^N is a solution of the reduced wave equation, (47) proves the first two assertions of Theorem 1. The last assertion of the theorem follows from (47) for a point on B and from the following inequality, which is implied by (9) and (10):

$$(4.48) \qquad |u_o - u_o^N| \leq \begin{cases} 0 & N \geq M \\ \sum_{N+1}^{M} \left| \frac{a_n}{k^n} \right| = O(k^{-N-1}) & N < M \end{cases} .$$

Proof of Theorem 4.2. Since $\Delta u + k^2 u = 0$, (44) becomes

$$(4.49) \qquad \Delta u^N + k^2 u^N = \frac{1}{(ik)^N} e^{ik\psi} \Delta \sigma_N .$$

From Lemmas 1 and 2, we see that we may apply the corollary of Theorem 1.3 to u^N. This gives

$$(4.50) \qquad \begin{aligned} 4iu^N(P) = &\int_B \left[\frac{\partial u^N}{\partial n} H_o^{(1)}(k|P-Q|) - u^N \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|) \right] ds \\ &- \frac{1}{(ik)^N} \iint_D e^{ik\psi} \Delta \sigma_N H_o^{(1)}(k|P-Q|) dS. \end{aligned}$$

In the proof of Lemma 3 we pointed out that the domain integral in (41) was bounded by $O(k^{-2/5})$. The same estimate applies to the domain integral in (50), and this implies Theorem 2.

Proof of Theorem 4.3. In (50), let P approach B. In the limit we have for P = B

$$5iu^N(P) - \int_B \frac{\partial u^N}{\partial n} H_o^{(1)}(k|P-Q|)ds + \int_B u^N \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|)dS$$

$$= -\frac{1}{(ik)^N} \iint_D e^{ik\psi} \triangle \sigma_N H_o^{(1)}(k|P-Q|)dS. \tag{4.51}$$

Using (9) and (10), we can write this as

$$5iu(P) - \int_B \frac{\partial u^N}{\partial n} H_o^{(1)} ds + \int_B u \frac{\partial}{\partial n} H_o^{(1)} ds$$

$$= -\frac{1}{(ik)^N} \iint_D e^{ik\psi} \triangle \sigma_N H_o^{(1)} dS + 5ie^{ik\psi} \sum_{N+1}^{M} \frac{a_n}{(ik)^n} \tag{4.52}$$

$$+ \int_B e^{ik\psi} \sum_{N+1}^{M} \frac{a_n}{(ik)^n} \frac{\partial}{\partial n} H_o^{(1)} ds.$$

If $N \geq M$, the last two terms in the right member are zero. Thus (52) together with the estimate for the domain integral derived in Lemma 3 imply

$$\left| 5iu(P) - \int_B \frac{\partial u^N}{\partial n} H_o^{(1)} ds + \int_B u \frac{\partial}{\partial n} H_o^{(1)} ds \right| \leq O(k^{-N-2/5}). \tag{4.53}$$

This proves the theorem in this case. For $N < M$, we must show that the last two terms of the right member of (52) are bounded by $O(k^{-N-1/3})$. The first of these terms is $O(k^{-N-1})$, so that we need only consider the integral. We have

$$\left| \int_B e^{ik\psi} \sum_{N+1}^{M} \frac{a_n}{(ik)^n} H_o^{(1)} ds \right| \leq O(k^{-N-1}) \int_B \left| \frac{\partial}{\partial n} H_o^{(1)} \right| ds. \tag{4.54}$$

By the methods used by Sternberg[17], we can show that the integral in the right member of (54) is bounded by $O(k^{2/3})$. This and (54) complete the proof of the theorem.

Proof of Theorem 4.4. From Theorem 1 we have for $\operatorname{Im} k > 0$

$$u^N = u + R^N + O(|k|^{-N-2/5}), \tag{4.55}$$

and

$$|R_o^N| \leq O(|k|^{-N-2/5}). \tag{4.56}$$

R^N as given in (45) is a sum of layers. A computation with (45) shows that R^N satisfies the radiation condition. Thus by Theorem 1.3,

$$R^N(P) = \frac{i}{4}\int_B \left[H_o^{(1)}(k|P-Q|)\,\frac{\partial R^N}{\partial n} - R^N \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|)\right] ds. \tag{4.57}$$

From this we get

$$\begin{aligned} \operatorname{lub}|R^N(P)| \leq \frac{1}{4}\Bigg[& \operatorname*{lub}_{P\varepsilon B} |R^N(P)| \int_B \left| \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|)\right| ds \\ & + \sqrt{\int_B |H_o^{(1)}|^2 ds}\ \sqrt{\int_B \left| \frac{\partial R^N}{\partial n}\right|^2 ds}\ \Bigg] . \end{aligned} \tag{4.58}$$

Let C(B) be larger than the right member of (3.3). Then the estimate of Lemma 3.2 is applicable. Inserting this estimate in (58), we obtain

$$\begin{aligned} \operatorname{lub}|R^N(P)| \leq \frac{1}{4}\Bigg[& O(|k|^{-N-\,2/5}) \int_B \left| \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|)\right| ds \\ & + \sqrt{\int_B |H_o^{(1)}|^2 ds}\ \sqrt{\frac{1}{1-2\varepsilon^2}}\ \sqrt{\frac{|k|^2}{\varepsilon^2}\left(1 + \frac{|k|^2}{2(\operatorname{Im}k)^2}\right)\int_B |R^N|^2 ds + \int_B \left|\frac{\partial R^N}{\partial s}\right|^2 ds}\ \Bigg]. \end{aligned} \tag{4.59}$$

To estimate $\left|\frac{\partial R^N}{\partial s}\right|$, we proceed as follows. Setting $f = (-i)^N e^{ik\Psi} \triangle \sigma_N$ in (41), we obtain

$$4iw^N(P) = 4ik^N R^N(P) - \iint_D (-i)^N e^{ik\Psi} \triangle \sigma_N H_o^{(1)}(k|P-Q|)\,dS. \tag{4.60}$$

From (43), (9) and (10), (60) becomes

$$R^N(P) = -\, e^{ik\Psi} \sum_{N+1}^{M} \frac{a_n}{(ik)^n} + \frac{(-i)^N}{4ik^N} \iint_D e^{ik\Psi} \triangle \sigma_N H_o^{(1)}(k|P-Q|)\,dS. \tag{4.61}$$

Since by Lemma 1, $|\triangle \sigma_N|$ is bounded, we obtain from (61)

$$\left| \frac{d}{ds} R^N(P)\right| \leq O(|k|^{-N}) + O(|k|^{-N}) \iint_D \left| \frac{\partial}{\partial s} H_o^{(1)}(k|P-Q|)\right| dS. \tag{4.62}$$

Using the technique of Sternberg's estimate[17], we can show that

$$\begin{aligned} & \int_B \left| \frac{\partial}{\partial n} H_o^{(1)}\right| ds \leq O(|k|^{2/3}) \\ & \int_B |H_o^{(1)}|^2 ds \leq O(|k|^{-2/3}) \end{aligned} \tag{4.63}$$

and

$$(4.64)\qquad \iint\limits_{D} \left| \frac{\partial}{\partial s} H_o^{(1)} \right| dS \;\leq\; O(|k|^{3/5}).$$

Now insert (64) into (62). Insert this result and (63) into (59). We obtain

$$\begin{aligned}
\operatorname{lub}|R^N(P)| \;&\leq\; O(|k|^{-N-\,2/5})\,O(|k|^{2/3}) \\
(4.65)\qquad &+ O(|k|^{-1/3})\sqrt{\frac{O(|k|^{4})O(|k|^{-2N-\,4/5})}{(\mathrm{Im}\ k)^2} + O(|k|^{-2N})O(|k|^{6/5})} \\
&\leq O(|k|^{-N+\,4/15}) + \frac{O(|k|^{-N+\,19/15})}{(\mathrm{Im}\ k)^2} = \frac{O(|k|^{-N+\,19/15})}{(\mathrm{Im}\ k)^2}.
\end{aligned}$$

Inserting this into (56), we obtain

$$(4.66)\qquad u^N = u + \frac{O(|k|^{-N+\,19/15})}{(\mathrm{Im}\ k)^2},$$

which proves the theorem.

Proof of Theorem 4.5. The proof is the same as that of Theorem 4, except that the estimate of Lemma 3.3 is used rather than the estimate of Lemma 3.2.

We now examine these results for the case of regular domains with infinite boundaries. Since the estimates of Section 3 are not known to be valid when B is infinite, Theorems 4 and 5 are not necessarily true in this case. An examination of the proofs of the remaining theorems of this section reveals that they still hold if the representation of the corollary of Theorem 1.3 for u^N and w^N is applicable when B is infinite (see (41), (50) and (51)). Theorem 3 requires an additional consideration which we will mention below. The corollary of Theorem 1.4, which provides this representation when B is infinite, requires that certain boundary integrals exist. By the remark preceding the corollary of Theorem 1.4, these integrals will converge and the proofs of Theorems 1, 2, and 3 will be valid for B infinite if

$$(4.67)\qquad \lim_{R\to\infty} R \max\left\{ |u_o|,\ \left|\frac{\partial u_o}{\partial n}\right|,\ |u_o^N|,\ \left|\frac{\partial u_o^N}{\partial n}\right| \right\} < \text{constant}.$$

Now in the proof of Theorem 3 in the case that $N < M$, we used (see (54))

$$(4.68)\qquad \int\limits_{B} \left| \sum_{N+1}^{M} a_n(Q) \frac{\partial}{\partial n} H_o^{(1)}(k|P-Q|) \right| ds \;\leq\; O(k^{2/3}).$$

(67) implies (68) for B infinite.

We formulate this discussion as

Theorem 4.6. Under the additional condition (67), Theorems 1, 2, and 3 are valid when B is infinite.

For $N \geq M$, we may make the following stronger statement.

Theorem 4.7. If the representations of (41), (50), and (51) are valid, then Theorems 1, 2, and 3 are true when B is infinite.

Finally, we examine the results for the case of non-regular domains.

A domain D which is non-regular relative to ϕ may be divided into a regular domain R and a non-regular domain N; then $D = R \cup N$. In the figure, the exterior of a circle with $\phi = x$, $a_o = 1$, $a_n = 0$, $n \geq 1$ is decomposed in such a manner. On the left side of the circle we choose the upper sign in (12), while on the right side we choose the lower sign. The caustic of the rays from the left side is indicated in the figure. The caustic of the rays from the right side is the point at $x = -\infty$. The common boundary of R and N is formed by the two rays L. The common boundary is chosen in this manner in order to make $\dot{\xi}$, $\dot{\eta}$, $\ddot{\xi}$, $\ddot{\eta}$ bounded in R.

Let $B_R = R \cap B$. Let $T = L \cup B_R$ be the boundary of R.

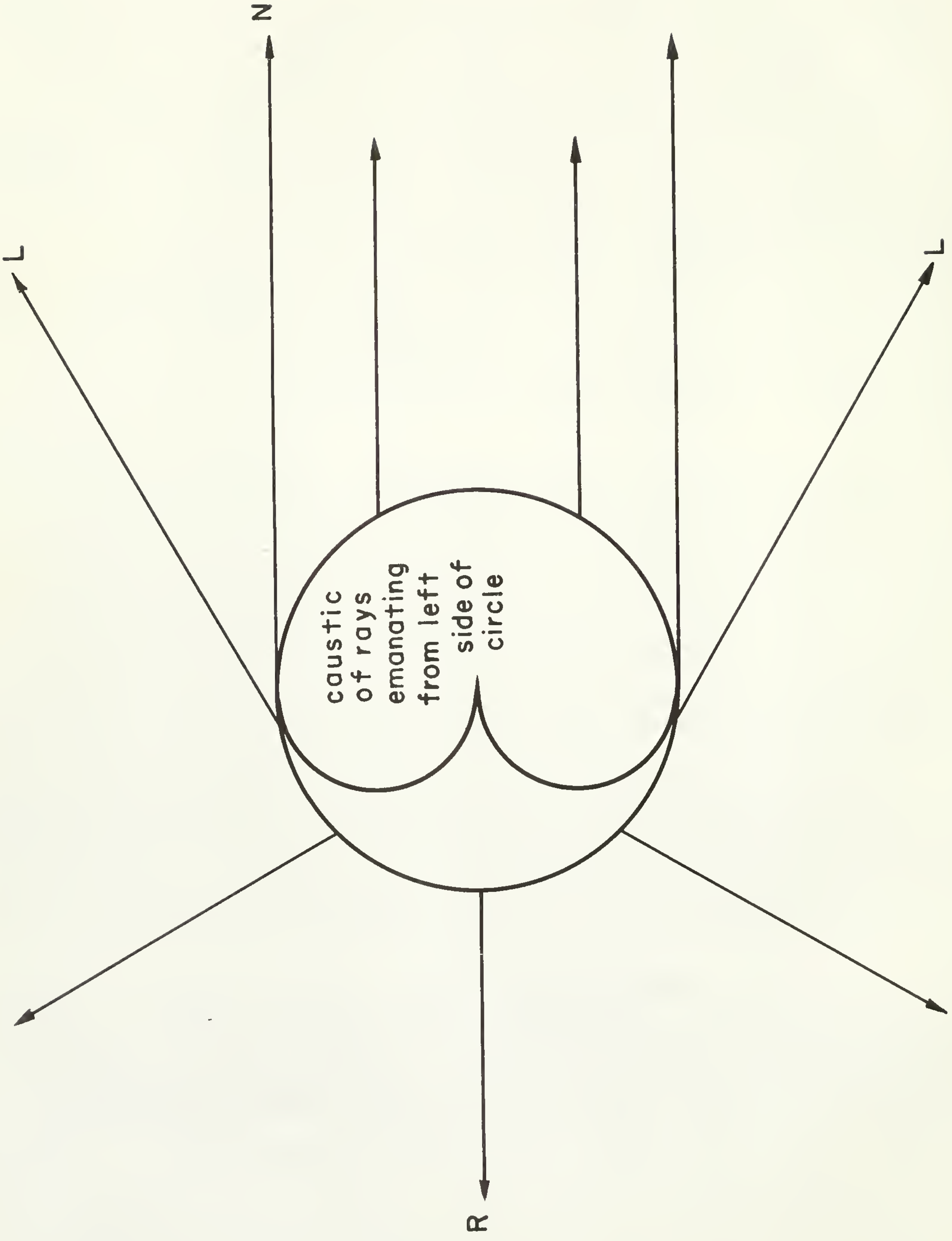
Our theorems are inapplicable to N because the caustic of the rays in N is at $x = -\infty$. However the domain R with boundary T falls into the case of regular domains with infinite boundaries considered above, except for some contingencies. The first of these concerns the representations used in (41), (50), and (51). The integrals over B are now replaced by integrals over T. Some assumption must be made about the convergence of these integrals. The restriction of (67) would suffice but is artificial since L is in the interior of D, the domain of definition of u.

In Theorem 1, the conclusion that the boundary values of the remainder solution R^N are less than $O(k^{-N-2/5})$ must be replaced by the conclusion that the values of R^N along B_R are of this order of magnitude.

With these alterations Theorems 6 and 7 are valid for the domain R.

Acknowledgment

The author is grateful to the members of the faculty and staff of the Graduate School of Mathematics for their many kindnesses. In particular he is grateful to Professor Joseph B. Keller for his advice and criticism during the preparation of this report.


N
L
L
caustic
of rays
emanating
from left
side of
circle
R

References

[1] Atkinson, F.V. - On Sommerfeld's radiation condition; The Philosophical Magazine, 40, 645 (1949).

[2] Bateman Manuscript Project - Vol. 2, McGraw Hill, 1953.

[3] Birkhoff, G.D. - Quantum mechanics and asymptotic series; Bull. Amer. Math. Soc., 39, 681 (1933).

[4] Bers, L. - Local behavior of solutions of general linear elliptic equations; Comm. on Pure and Applied Math., 8, 473 (1955).

[5] Hörmander, L. - Inequalities between normal and tangential derivatives of harmonic functions; unpublished.

[6] Jones, D.S. - The eigenvalues of $\nabla^2 u + \lambda u = 0$ when the boundary conditions are given on semi-infinite domains; Proc. Cambridge Philosophical Society, 49, 668 (1953).

[7] Keller, J.B., Lewis, R.M., and Seckler, B.D. - Asymptotic solution of some diffraction problems; New York University, Institute of Mathematical Sciences, Research Report EM-81, June, 1955; also Comm. on Pure and Applied Math., 9, 207 (1956).

[8] Kline, M. - Asymptotic solution of linear hyperbolic partial differential equations; New York University, Institute of Mathematical Sciences, Research Report EM-48, Dec. 1952; also Journal of Rational Mechanics and Analysis, 3, 313 (1954).

[9] Levinson, N. - The first boundary-value problem for $\varepsilon \triangle u + A(x,y)u_x + B(x,y)u_y + C(x,y)u = D(x,y)$ for small ε; Annals of Math., 51, 428 (1950).

[10] Magnus, W. - Über Eindeutigkeitsfragen bei einer Randwertaufgabe von $\triangle u + k^2 u = 0$; Jahresbericht der Deutschen Mathematiker-Vereinigung, 52, 177 (1942).

[11] Magnus, W. - Fragen der Eindeutigkeit und des Verhaltens in Unendlichen für Lösungen von $\triangle u + k^2 u = 0$; Mathematische Seminar der Universität Hamburg, Vol. 16, 77 (1949).

[12] Magnus, W. and Oberhettinger, F. - Formeln und Sätze für die Speziellen Funktionen der Mathematischen Physik; Springer (1948).

[13] Peters, A. and Stoker, J.J. - A uniqueness theorem and a new solution for Sommerfeld's and other diffraction problems; Comm. on Pure and Applied Math., 7, 565 (1954).

[14] Rellich, F. - Über das asymptotischen Verhalten der Lösungen von $\Delta u + \lambda u = 0$ in unendlichen Gebieten; Jber. Deut. Math. Verein., 53, 57 (1943).

[15] Weber, H.M. - Die partiellen Differentialgleichungen der mathematischen Physik, nach Riemann's 5te Auflage; F. Ziewer und Son, 1919, Vol. 2, p. 281.

[16] Sommerfeld, A. - Die Greensche Funktion der Schwingungsgleichung; Jber. Deut. Math. Verein, 21, 309 (1912).

[17] Sternberg, W. - Über die Asymptotische Integration partieller Differentialgleichungen mit Parameter (2 parts); Math. Annalen, 86, 140, 280 (1922).

[18] Trjitzinsky, W.J. - Analytic theory of parametric linear partial differential equations; Recuiel Mathematic, 15, 179 (1944).

[19] Vishik, M. - On a certain inequality for the bounds of the values of harmonic functions in a sphere; (Russian) Uspekhi Matematicheskisk Nauk, 16, 2(42), p. 165 (1951).

[20] Wasow, W. - Asymptotic solution of boundary-value problems for $\Delta u + \lambda \frac{\partial}{\partial x} u = \lambda f(x,y)$; Duke Math. Journal, 11, 405 (1944).

[21] Watson, G.N. - Theory of Bessel functions; Macmillan Co., New York, 1948, 2nd Ed.

[22] Wilcox, C.H. - A generalization of theorems of Rellich and Atkinson; Proc. Amer. Math. Soc., 7, 271 (1956).

DISTRIBUTION LIST FOR RESEARCH REPORTS

Contract No. AF 18(600)-367

(ONE copy unless otherwise noted)

Commandant
Headquarters Air University
Maxwell Air Force Base, Alabama

(3)Commander
European Office, ARDC
60 Rue Ravenstein
Brussels, Belgium

Applied Mathematics & Statistics Lab.
Stanford University
Stanford, California

Department of Mathematics
University of California
Berkeley, California

Commander
Air Force Flight Test Center
ATTN: Technical Library
Edwards Air Force Base, California

The Rand Corporation
Technical Library
1700 Main Street
Santa Monica, California

Director of Advanced Studies
Air Force Office of Scientific Research
Post Office Box 2035
Pasadena 2, California

Commander
Western Development Division
ATTN: WDSIT
P.O. Box 262
Inglewood, California

Department of Mathematics
Yale University
New Haven, Connecticut

Chief of Naval Operations
Department of the Navy
ATTN: OPO3EG; Mr. Arthur Kaufman
Washington 25, D.C.

Human Factors Operations Research Labs.
Air Research and Development Command
Bolling Air Force Base
Washington 25, D.C.

(2)Office of Naval Research
Department of the Navy
ATTN: Code 432
Washington 25, D.C.

Department of Commerce
Office of Technical Services
Washington 25, D.C.

Director National Security Agency
ATTN: Dr. H.H. Campaigne
Washington 25, D.C.

Library
National Bureau of Standards
Washington 25, D.C.

National Applied Mathematics Labs.
National Bureau of Standards
Washington 25, D.C.

Headquarters, USAF
Director of Operations
ATTN: Operations Analysis Division, AFOOP
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRDB
Washington 25, D.C.

(2)Commander
Air Force Office of Scientific Research
ATTN: SRE
Washington 25, D.C.

National Science Foundation
1520 "H" St., N.W.
Washington 25, D.C.

Commander
Air Force Armament Center
ATTN: Technical Library
Eglin Air Force Base, Florida

Commander
Air Force Missile Test Center
ATTN: Technical Library
Patrick Air Force Base, Florida

Department of Mathematics
Northwestern University
Evanston, Illinois

Institute for Air Weapons Research
Museum of Science and Industry
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Chicago
Chicago 37, Illinois

Department of Mathematics
University of Illinois
Urbana, Illinois

Department of Mathematics
Purdue University
Lafayette, Indiana

Institute for Fluid Dynamics
and Applied Mathematics
University of Maryland
College Park, Maryland

Mathematics and Physics Library
The Johns Hopkins University
Baltimore, Maryland

Commander
HQ Air Research and Development Command
ATTN: Technical Library
P.O. Box 1395
Baltimore 3, Maryland

Department of Mathematics
Harvard University
Cambridge 38, Massachusetts

Department of Mathematics
Massachusetts Institute of Technology
Cambridge, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Geophysics Research Library
L.G. Hanscom Field
Bedford, Massachusetts

Commander
Air Force Cambridge Research Center
ATTN: Electronics Research Library
L.G. Hanscom Field
Bedford, Massachusetts

Dr. Y.W. Chen
2645 Stuart Street
Berkeley, California

Department of Mathematics
Wayne University
Detroit 1, Michigan

Willow Run Research Center
University of Michigan
Ypsilanti, Michigan

Department of Mathematics
Folwell Hall
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Institute of Technology
Engineering Building
University of Minnesota
Minneapolis, Minnesota

Department of Mathematics
Washington University
St. Louis 5, Missouri

Department of Mathematics
University of Missouri
Columbia, Missouri

Linda Hall Library
ATTN: Mr. Thomas Gillis
Document Division
5109 Cherry Street
Kansas City 10, Missouri

Commander
Strategic Air Command
ATTN: Operations Analysis
Offutt Air Force Base
Omaha, Nebraska

The James Forrestal Res. Center Lib.
Princeton University
Princeton, New Jersey

Library
Institute for Advanced Study
Princeton, New Jersey

Department of Mathematics
Fine Hall
Princeton University
Princeton, New Jersey

Commander
Holloman Air Development Center
ATTN: Technical Library
Holloman Air Force Base, New Mexico

Commander
Air Force Special Weapons Center
ATTN: Technical Library
Kirtland Air Force Base
Albuquerque, New Mexico

Prof. J. Wolfowitz
Mathematics Department
White Hall
Cornell University
Ithaca, New York

Department of Mathematics
Syracuse University
Syracuse, New York

New York University
Institute of Mathematical Sciences
Division of Electromagnetic Research
ATTN: Professor M. Kline
25 Waverly Place
New York 3, N.Y.

Department of Mathematics
Columbia University
ATTN: Professor B.O. Koopman
New York 27, N.Y.

Department of Mathematical Statistics
Fayerweather Hall
ATTN: Dr. Herbert Robbins
Columbia University
New York 27, N.Y.

Commander
Rome Air Development Center
ATTN: Technical Library
Griffiss Air Force Base
Rome, New York

Institute for Aeronautical Sciences
ATTN: Librarian
2 East 64th Street
New York 21, N.Y.

Institute of Statistics
North Carolina State College of A& E
Raleigh, North Carolina

Department of Mathematics
University of North Carolina
Chapel Hill, North Carolina

(2)Office of Ordnance Research
Box CM
Duke Station
Durham, North Carolina

Department of Mathematics
Duke University
Duke Station
Durham, North Carolina

Commander
Air Technical Intelligence Center
ATTN: ATIAE-4
Wright-Petterson Air Force Base, Ohio

Commander
Wright Air Development Center
ATTN: Technical Library
Wright-Patterson Air Force Base, Ohio

(2)Commander
Wright Air Development Center
ATTN: ARL Technical Library, WCRR
Wright-Patterson Air Force Base, Ohio

(2)Commandant
USAF Institute of Technology
ATTN: Technical Library, MCLI
Wright-Patterson Air Force Base, Ohio

(10)Chief, Document Service Center
Armed Services Technical Information Agency
Knott Building
Dayton 2, Ohio

Department of Mathematics
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Department of Mathematics
University of Pennsylvania
Philadelphia, Pennsylvania

Commander
Arnold Engineering Development Center
ATTN: Technical Library
Tullahoma, Tennessee

Defense Research Laboratory
University of Texas
Austin, Texas

Department of Mathematics
Rice Institute
Houston, Texas

Commander
AF Personnel & Training Research Center
ATTN: Technical Library
Lackland Air Force Base
San Antonio, Texas

Department of Mathematics
University of Wisconsin
Madison, Wisconsin

Mathematics Research Center
ATTN: R.E. Langer
University of Wisconsin
Madison, Wisconsin

Naval Electronics Laboratory
San Diego 52, California

Research Laboratory of Electronics
Massachusetts Institute of Technology
Rm. 20B-221
Cambridge 39, Massachusetts

Signal Corps Engineering Labs.
Technical Documents Center
Evans Signal Laboratory
Belmar, New Jersey

Commanding General
Signal Corps Engineering Labs.
ATTN: Technical Reports Library
Fort Monmouth, New Jersey

Chief of Staff
HQ United States Air Force
Pentagon, Washington 25, D.C.
ATTN: AFMRD - 5

Office of Chief Signal Officer
Signal Plans and Operations Division
Com. Liaison Br., Radio Prop. Section
The Pentagon, Washington 25, D.C.
ATTN: SIGOL - 2, Rm. 20

Library
Boulder Laboratory
National Bureau of Standards
Boulder, Colorado

Naval Research Laboratory
Washington 25, D.C.
ATTN: Technical Data Section

Air Force Development Field Rep.
Naval Research Laboratory
Code 1110
Washington 25, D.C.

Ballistics Research Laboratory
Aberdeen Proving Ground
Aberdeen, Maryland
ATTN: Dr. Keats Pullen

Massachusetts Institute of Technology
Lincoln Laboratory
P.O. Box 390
Cambridge 39, Massachusetts
ATTN: Dr. T.J. Carroll

Mr. Keeve Siegel
Willow Run Research Center
Willow Run Airport
Ypsilanti, Michigan

Willow Run Research Center
University of Michigan
Willow Run Airport
Ypsilanti, Michigan
ATTN: Dr. C.L. Dolph

Director
Naval Research Laboratory
Code 3480
Washington 25, D.C.
ATTN: Dr. L.C. van Atta

Director
Naval Research Laboratory
Washington 25, D.C.
ATTN: Mr. Robert E. Roberson

National Bureau of Standards
Computation Laboratories
Washington 25, D.C.
ATTN: Dr. John Todd

Dr. A.G. McNish
National Bureau of Standards
Washington 25, D.C.

Dr. Derrick H. Lehmer
Department of Mathematics
University of California
Berkeley, California

Division of Electrical Engineering
Electronics Research Laboratory
University of California
Berkeley 4, California
ATTN: Dr. Samuel Silver

Dr. Joseph Kaplan
Department of Physics
University of California
Los Angeles, California

Dr. A. Erdelyi
California Institute of Technology
1201 E. California Street
Pasadena, California

Dr. Robert Kalaba
Electronics Division
Rand Corporation
Santa Monica, California

Stanford Research Institute
Stanford, California
ATTN: Dr. J.V.N. Granger
Head, Radio Systems Laboratory

Dr. Vic Twersky
Electronics Defense Laboratory
Box 205
Mountain View, California

Hughes Aircraft Company
Research and Development Library
Culver City, California
ATTN: M. Bodner

Georgia Institute of Technology
Engineering Experimental Station
Atlanta, Georgia
ATTN: Dr. James E. Boyd

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Mr. F.T. McClure

Dr. Donald E. Kerr
Department of Physics
The Johns Hopkins University
Baltimore 18, Maryland

Dr. A. Weinstein
Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland

Professor J.A. Pierce
Harvard University
Cambridge 38, Massachusetts

Dr. F.M. Wiener
Bolt Beranek and Newman Inc.
16 Eliot Street
Cambridge 38, Massachusetts

Mrs. Marjorie L. Cox, Librarian
Technical Reports Collection
Room 303A, Pierce Hall
Harvard University
Cambridge 38, Massachusetts

Dr. Arthur A. Oliner
Microwave Research Institute
Polytechnic Institute of Brooklyn
55 Johnson Street
Brooklyn, N.Y.

Professor J.H. Mulligan
School of Engineering
New York University
New York, N.Y.

Dr. H.G. Booker
School of Electrical Engineering
Cornell University
Ithaca, N.Y.

Professor Bernard Epstein
Department of Mathematics
University of Pennsylvania
Philadelphia 4, Pennsylvania

Professor Albert Heins
Carnegie Institute of Technology
Pittsburgh, Pennsylvania

Professor Fred A. Ficken
University of Tennessee
Knoxville, Tennessee

Miss Barbara C. Grimes, Librarian
Federal Communications Commission
Washington 25, D.C.

Carnegie Institute of Washington
Department of Terrestrial Magnetism
5241 Broad Branch Road, N.W.
Washington 15, D.C.
ATTN: Library

Mathematical Reviews
80 Waterman Street
Brown University
Providence, Rhode Island

Mr. Martin Katzin
154 Fleetwood Terrace
Silver Spring, Maryland

Prof. B.H. Bissinger
Lebanon Valley College
Annville, Pennsylvania

Applied Physics Laboratory
The Johns Hopkins University
8621 Georgia Avenue
Silver Spring, Maryland
ATTN: Dr. B.S. Gourary

Dr. Charles H. Papas
Dept. of Electrical Engineering
California Institute of Technology
Pasadena, California

Dr. Rodman Doll
209A Emmet Street
Ypsilanti, Michigan

California Institute of Technology
Pasadena 4, California
ATTN: Mr. Calvin Wilcox

Technical Director
Combat Development Dept.
Army Electronic Proving Ground
Fort Huachuca, Arizona

National Bureau of Standards
Boulder, Colorado
ATTN: Dr. W.R. Callet

Mrs. A.M. Gray
Engineering Library, Plant 5
Grumman Aircraft Corp.
Bethpage, L.I., N.Y.

Professor Bernard Friedman
55 Hilltop Avenue
New Rochelle, N.Y.

Dr. Jane Scanlon
224-C3 67th Avenue
Bayside 64, N.Y.

Dr. Solomon L. Schwebel
3689 Louis Road
Palo Alto, California

Dr. Bernard Lippmann
Eastern Technical Associates
38 Main Street
Hastings-On-Hudson, N.Y.

Engineering Library
University of California
405 Hilgard Avenue
Los Angeles 24, California

University of Minnesota
The University Library
Minneapolis 14, Minnesota
ATTN: Exchange Division

Lincoln Laboratory
Massachusetts Institute of Technology
P.O. Box 73
Lexington 73, Massachusetts
ATTN: Dr. Shou Chin Wang, Rm. C-351

Technical Research Group
17 Union Square West
New York 3, N.Y.
ATTN: Dr. L. Goldmuntz

Hoffman Laboratories Inc.
Advanced Development Section
3761 South Hill Street
Los Angeles 7, California
ATTN: Dr. Richard B. Barrar

Institute of Fluid Dynamics and
Applied Mathematics
University of Maryland
College Park, Maryland
ATTN: Dr. Elliott Montroll

Brandeis University
Waltham, Massachusetts
ATTN: Library

General Electric Company
Microwave Laboratory
Electronics Division
Stanford Industrial Park
Palo Alto, California
ATTN: Library

Dr. John B. Smyth
Smyth Research Associates
3930 4th Avenue
San Diego 3, California

Dr. Georges G. Weill
Electrical Engineering
California Institute of Technology
Pasadena, California

Naval Research Laboratory
Washington 25, D.C.
ATTN: Dr. Henry J. Passerini
Code 5278 A

Dr. George Kear
10585 N. Stelling Road
Cupertino, California

California Institute of Technology
Electrical Engineering
Pasadena, California
ATTN: Dr. Zohrab A. Kaprielian

Prof. Nathan Marcuvitz
Microwave Research Institute
Brooklyn Polytechnic
55 Johnson Street
Brooklyn, N.Y.

Dr. Lester Kraus
4935 WhitehavenWay
San Diego 10, California

Dr. Jerry Shmoys
Dept. of Electrical Engineering
Brooklyn Polytechnic
85 Livingston Street
Brooklyn, N.Y.

Dr. I. Kolodner
Dept. of Mathematics
University of New Mexico
Albuquerque, New Mexico

Dr. Harry Hochstadt
W.L. Maxon
460 W. 34th Street
New York, N.Y.

Hycon Eastern Inc.
75 Cambridge Parkway
Cambridge, Massachusetts
ATTN: Seymour Stein

Mr. K.S. Kelleher, Section Head
Melpar, Inc.
3000 Arlington Boulevard
Falls Church, Virginia

Department of Engineering
Brown University
Providence, R.I.
ATTN: Dr. V.M. Papadopoulos

Professor C.A. Woonton, Director
Eaton Electronics Research Laboratory
McGill University
Montreal, Canada

Dr. D.S. Jones
University of North Staffordshire
Keele, Staffordshire
England

Science Abstracts
Institute of Electrical Engineers
Savoy Place
London W.C. 2, England
ATTN: R.M. Crowther

Dr. Jean-Claude Simon
Centre de Recherches techniques
Compagnie generale de T.S.F.
Paris 19, France

Professor A. van Wijngaarden
Mathematisch Centrum
2^e Boerhaavestraat 49
Amsterdam-Zuid, Holland

Dr. C.J. Bouwkamp
Philips Research Laboratories
Eindhoven, Netherlands

Dr. E.T. Copson
Department of Mathematics
United College
University of St. Andrews
St. Andrews, Scotland

Universite de Paris
Cabinet du Department
des Sciences Mathematiques
Institut Henri Poincare
11 Rue Pierre Curie
Paris 5^e, France

Dr. W. Elwyn Williams
24 Manton Drive
Luton
Bedfordshire, England

University of Cambridge
Cavendish Laboratory
Cambridge, England
ATTN: Prof. Philip Clemmow

Technische Hogeschool
Instituut voor Toegepaste Wiskunde
Jaffalaan 162
Delft, Holland
ATTN: Prof. R. Timman

Instituto de Fisica Teorica
Rue Pamplona, 145
S. Paulo, Brasil

Dr. H. Bremmer
Philips Research Laboratories
Eindhoven, Netherlands

Electrotechnical Laboratory
Nagata-cho
Chiyoda-ku
Tokyo, Japan
ATTN: Library

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
[illegible] New York 3, N.Y.

NYU BR21
BR-21

NYU BR-21
BR-21
AUTHOR Miranker
TITLE The asymptotic theory of
sol. of $\Delta u+k^2 u=0$

NYU BR-21
BR-21 Miranker
The asymptotic theory of
solutions of $\Delta u+k^2 u=0$

Date Due

AUG 5 '59
DEC 15 '59
JAN 30 '60
NOV 1 '60
NOV 23 '60
JAN 3 '61
MAR 17 '61
FEB 28 '63
AUG 28 '64
DEC [illegible] '66
JUN 14 '66
MAR 10 '67
SEP 29 1961
DEC [illegible]
OCT 26 [illegible]

PRINTED IN U. S. A.